

Self-motions of pentapods with linear platform

Georg Nawratil^a, Josef Schicho^b

^aInstitute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstrasse 8-10/104, 1040 Vienna, Austria

^bJohann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenberger Strasse 69, 4040 Linz, Austria

Abstract

We give a full classification of all pentapods with linear platform possessing a self-motion beside the trivial rotation about the platform. Recent research necessitates a contemporary and accurate re-examination of old results on this topic given by Darboux, Mannheim, Duporcq and Bricard, which also takes the coincidence of platform anchor points into account. For our study we use bond theory with respect to a novel kinematic mapping for pentapods with linear platform, beside the method of singular-invariant leg-rearrangements. Based on our results we design pentapods with linear platform, which have a simplified direct kinematics concerning their number of (real) solutions.

Keywords: Pentapod, Borel Bricard Problem, Bond Theory, Self-Motion, Direct Kinematic, Kinematic Mapping

1. Introduction

The geometry of a pentapod (see Fig. 2a) is given by the five base anchor points M_i with coordinates $\mathbf{M}_i := (A_i, B_i, C_i)^T$ with respect to the fixed system Σ_0 and by the five collinear platform anchor points m_i with coordinates $\mathbf{m}_i := (a_i, 0, 0)^T$ with respect to the moving system Σ (for $i = 1, \dots, 5$). All pairs (M_i, m_i) of corresponding anchor points are connected by SPS-legs (or alternatively SPU-legs with aligned universal joints; cf. [1, Fig. 1]), where only the prismatic joints are active.

If the geometry of the manipulator is given, as well as the lengths of the five pairwise distinct legs, a pentapod has generically mobility 1 according to the formula of Grübler. In the discussed case of pentapods with linear platform the degree of freedom corresponds to the rotation about the carrier line \mathbf{p} of the five platform anchor points. This rotational motion is irrelevant for applications with axial symmetry as e.g. 5-axis milling, spot-welding, laser or water-jet engraving/cutting, spray-based painting, etc. (cf. [2, 3]). Therefore these mechanisms are of great practical interest. In this context configurations should be avoided, where the manipulator gains an additional uncontrollable mobility (beside the rotational motion around \mathbf{p}), which is referred as self-motion within this article. Before we give a review on pentapods possessing these special motions in Section 1.2, we repeat a few basics in geometry, which are essential for the understanding of the paper.

1.1. Geometric basics

We consider the projective closure of the Euclidean 3-space, which means that we add a point at infinity to each line, which is a so-called ideal point. Moreover two lines are parallel or coincide if and only if they have the same ideal point. The set of ideal points of a pencil of lines (cf. Fig. 1a) form a so-called ideal line of the carrier plane of this pencil. Again two planes are parallel or coincide if and only if they have the ideal line in common. The set of ideal points of a bundle of lines (cf. Fig. 1b) constitute the so-called ideal plane. Summed up we can say that we obtain the projective closure by addition of the ideal plane. Points, lines and planes, which are no ideal elements are called finite.

We can also introduce projective point coordinates by homogenizing the coordinates (x, y, z) of a finite point by $(1 : x : y : z)$. The ideal point of a finite line in direction (u, v, w) has the coordinates $(0 : u : v : w)$. Now we can define a regular/singular projectivity (projective mapping) by multiplication of the projective point coordinates with a regular/singular 4×4 matrix. If the set of ideal points is mapped onto itself, then the projectivity is called an affinity (affine transformation).

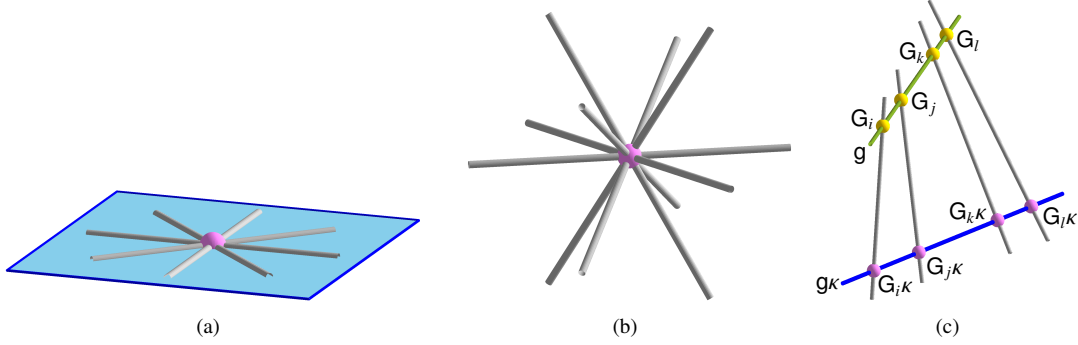


Figure 1: (a) A pencil of lines is the 1-parametric set of all coplanar lines through a common point, which is the co-called vertex of the pencil. (b) A bundle of lines is the 2-parametric set of all lines through a common point, which is the co-called vertex of the bundle. (c) A regulus of lines.

Under a regular projectivity a line g is mapped onto a line g_K due to the linearity of the mapping. The set of lines $[G_i, G_{iK}]$ with $G_i \in g$ is called a regulus of lines (cf. Fig. 1c) if g and g_K are skew. In this context it should be noted that the cross-ratio (CR) of four collinear points (cf. [4, page 20]) is invariant under a regular projectivity; i.e. $CR(G_i, G_j, G_k, G_l) = CR(G_{iK}, G_{jK}, G_{kK}, G_{lK})$.

Conic sections are well-known geometric objects. If the ideal line of the carrier plane of the conic section touches the conic, then it is a parabola. If there are conjugate complex (resp. two real) intersection points, we get an ellipse (resp. a hyperbola). The conjugate complex intersection points of a circle with the ideal line are the co-called cyclic points of the circle's carrier plane.

A further geometric object used within the article at hand is a so-called cubic ellipse. According to [5] a cubic ellipse is a space curve of degree 3, which intersects the ideal plane in one real and two conjugate complex ideal points¹. If the latter ones are the cyclic points of a plane (not) orthogonal to the direction of the real ideal point, then the cubic ellipse is called straight cubic circle (skew cubic circle). For more details we refer to [6].

Finally we need the notation of a so-called Möbius transformation τ of the plane. If we combine the planar Cartesian coordinates (u, v) to a complex number $w := u + iv$, then $\tau(w)$ can be defined as a rational function of the form

$$\tau : w \mapsto \frac{z_1 w + z_2}{z_3 w + z_4}, \quad (1)$$

with complex numbers z_1, \dots, z_4 satisfying $z_1 z_4 - z_2 z_3 \neq 0$. Moreover it should be noted that τ maps straight lines onto straight lines or circles and that a Möbius transformation is uniquely defined by three pairwise distinct points w_1, w_2, w_3 and their pairwise distinct images.

1.2. Review, motivation and outline

The self-motions of pentapods with linear platform represent interesting solutions to the still unsolved problem posed by the French Academy of Science for the *Prix Vaillant* of the year 1904, which is also known as Borel-Bricard problem (cf. [7, 8, 9]) and reads as follows: "*Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths.*"

For the special case of five collinear points the Borel-Bricard problem seemed to be solved since more than 100 years, due to the following results (cf. [10, page 415]): If five points of a line have spherical trajectories then this property holds for all points of the line. The centers are located on a straight line (cf. Darboux [11, page 222]), a conic section (cf. Mannheim [12, pages 180ff]) or a straight cubic circle (cf. Duporcq [13]; see also Bricard [8, Chapter III]).

In a recent publication [14] the authors determined all pentapods with mobility 2, where neither all platform anchor points nor all base anchor points are collinear. As a side result of this study we obtained the following three designs of pentapods with linear platform possessing a self-motion of p:

¹Therefore the projection in direction of the real ideal point yields an ellipse.

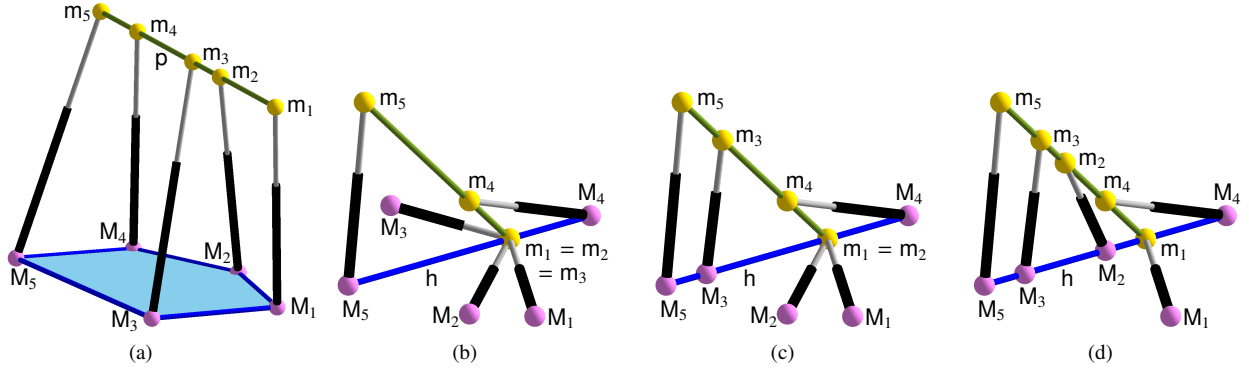


Figure 2: (a) Pentapod with linear platform p (and planar base). (b,c,d) The designs (α, β, γ) are illustrated in a pose of a self-motion.

- (α) $m_1 = m_2 = m_3$: The self-motion is obtained if $m_1 = m_2 = m_3$ is located on the line h spanned by M_4 and M_5 .
- (β) $m_1 = m_2$ and M_3, M_4, M_5 collinear: The self-motion is obtained if $m_1 = m_2$ is located on the line h spanned by M_3, M_4, M_5 .
- (γ) M_2, M_3, M_4, M_5 collinear: The self-motion is obtained if m_1 is located on the line h spanned by M_2, M_3, M_4, M_5 .

In all three cases, which are illustrated in Fig. 2, the following legs can be added without restricting the spherical self-motion: Every point of p can be connected with any point of the line h with exception of the point m_1 (= center of spherical motion), which can be linked with any point of the fixed 3-space.

This already shows that the cases (α, β, γ) are not covered by the above cited results of Darboux, Mannheim, Duporcq and Bricard, respectively. The reason for this is partially hidden in the formulation of the problem, as 100 years ago they studied the conditions for five points of a line to run on spherical trajectories, which already implies that the collinear points are pairwise distinct. Therefore they missed the cases (α, β) , but this still does not explain the absence of case (γ). All in all this shows the need of a contemporary and accurate re-examination of the old results, which also takes the coincidence of platform anchor points into account. This is done within the paper at hand, which is structured as follows:

In Section 1.3 we give a short review on architecturally singular pentapods as they imply further solutions to our problem. Based on the method of singular-invariant leg-replacements we determine in Section 2 all non-architecturally singular pentapods with linear platform and planar base possessing self-motions. The same method is used in Section 3 to distinguish five different types of pentapods with linear platform and non-planar base. In Section 4 we introduce the theory of bonds with respect to a novel kinematic mapping for pentapods with linear platform. This theory is used in Section 5 for the determination of all non-architecturally singular pentapods with linear platform and non-planar base possessing self-motions. Finally we use the presented results/methods to design pentapods with a linear platform, which have a simplified direct kinematics with respect to the number of (real) solutions (cf. Section 6).

1.3. Architecturally singular pentapods

A pentapod is called architecturally singular if in any pose of the platform the rank of its Jacobian matrix is less than five. This condition also has a line-geometric interpretation as the Jacobian is composed of the Plücker coordinates of the five carrier lines of the legs (cf. [4]). All architecturally singular pentapods are known (cf. [15, Theorem 3] under consideration of [16]) as well as their properties of being redundant and invariant under projectivities of the platform and the base. If we take additionally the collinearity of the platform into account (i.e. m_1, \dots, m_5 collinear) we end up with the following list:

Corollary 1. *If a pentapod with linear platform is architecturally singular, then it has one of the following designs ²:*

²After a possible necessary renumbering of anchor points.

1. $M_1 = M_2 = M_3$: The first three legs belong to a pencil of lines.
2. $m_1 = m_2 = m_3$ and M_1, M_2, M_3 are collinear: The first three legs belong to a pencil of lines.
3. M_1, \dots, M_4 are collinear and the following cross-ratio condition holds:

$$CR(m_1, m_2, m_3, m_4) = CR(M_1, M_2, M_3, M_4). \quad (2)$$

The first four lines belong to a regulus of lines. In the special case $m_1 = m_2$ and $M_3 = M_4$ the regulus splits up into two pencils of lines.

4. $m_1 = m_2 = m_3 = m_4$: The first four legs belong to a bundle of lines.
5. M_1, \dots, M_5 are collinear.
6. $m_1 = m_2 = m_3$ and $M_4 = M_5$.
7. $m_1 = m_2$ and $m_4 = m_5$. Moreover M_1, M_2, M_5 are collinear and M_3, M_4, M_5 are collinear.
8. m_1, \dots, m_5 are pairwise distinct, M_1, \dots, M_5 are coplanar and no three of them are collinear. Moreover there is a projective correspondence between the points m_i and M_i for $i = 1, \dots, 5$. For more details on this most complicated case we refer to [15, Item 10 of Theorem 3] and [17].
9. $m_4 = m_5$ holds and M_1, \dots, M_5 are coplanar, where M_1, M_2, M_3 are collinear. Moreover the following cross-ratio condition holds:

$$CR(m_1, m_2, m_3, m_4 = m_5) = CR(M_1, M_2, M_3, M), \quad (3)$$

with M denoting the intersection point of $[M_4, M_5]$ and the carrier line of M_1, M_2, M_3 .

In the cases (5-9) the five legs belong to a so-called congruence of lines (cf. [4, Section 3.2.1]).

Due to the above mentioned redundancy all these nine cases imply solutions to our problem, as they have a self-motion in each pose of p (over \mathbb{C}). Moreover, the architecturally singular cases 2,4,6,7,9 are also not covered by the old results due to the coincidence of platform anchor points. The remaining cases are discussed in more detail:

- Ad 3: All lines of the regulus can be added without restricting the self-motion. Therefore this case corresponds to the result of Darboux [11, page 222].
- Ad 8: The projective correspondence can be extended to all points of the linear platform p and therefore they are mapped onto a conic determined by M_1, \dots, M_5 . Now all legs connecting corresponding anchor points can be attached without changing the self-motion. This equals the solution given by Mannheim [12, pages 180ff].
- Ad 1: This case can be interpreted as a special case of Mannheim's solution, as following legs can be added without restricting the self-motion: Every point of p can be connected with $M_1 = M_2 = M_3$ with exception of the point m_i which can be linked with any point of the line $[M_1, M_i]$ for $i = 4, 5$. Therefore the conic splits up into the two lines $[M_1, M_4]$ and $[M_1, M_5]$.
- Ad 5: This trivial case can also be seen as a special case of Mannheim's solution, as the conic degenerates into the double counted carrier line of M_1, \dots, M_5 .

As all architecturally singular pentapods with linear platform are already known, we can restrict our study done in the remainder of the article to non-architecturally singular manipulators. Moreover, as the designs (α, β, γ) are not architecturally singular, we can make the following three additional assumptions in order to exclude these already known cases:

- (i) No three platform anchor points coincide.
- (ii) If two platform anchor points coincide, the remaining three base anchor points are not collinear.

(iii) No four base anchor points are collinear.

Definition 1. We define by \mathcal{P} the set of all non-architecturally singular pentapods with a linear platform, which fulfill the assumptions (i,ii,iii).

We split the determination of all elements of \mathcal{P} with self-motions in two parts with respect to the criterion if the base anchor points are coplanar (= planar pentapod; e.g. Fig. 2a,d) or not (= non-planar pentapod; e.g. Fig. 2b,c).

2. Planar pentapods of \mathcal{P} with self-motions

Within this section we prove the following theorem:

Theorem 1. A planar pentapod of \mathcal{P} has a self-motion only in the following case: There exists an orthogonal-projection π_ε of the base plane ε and an orthogonal-projection π_p of p in a way that the projected point sets are congruent. In this case p can perform a circular translation.

For the proof of this theorem the following preparatory work has to be done:

Lemma 1. The anchor points of a planar pentapod with a linear platform, which fulfills the assumptions (i,ii,iii), can always be relabeled in a way that the following conditions hold:

$$M_1 \neq M_2, \quad M_1, M_2, M_3 \text{ not collinear}, \quad M_1, M_2, M_4 \text{ not collinear}, \quad m_3 \neq m_4.$$

PROOF: If two platform anchor points coincide we denote them with $m_4 = m_5$. Then due to assumption (ii) M_1, M_2, M_3 are not collinear. Due two assumption (iii) one of the remaining two base points is not on the line spanned by M_1 and M_2 . We denote this point by M_4 and we are done.

Now we discuss the case where all five platform anchor points are pairwise distinct:

- ★ If two base points coincide³ then we denote them with $M_3 = M_4$. Due to assumption (iii) there are at least two further base points which span together with $M_3 = M_4$ a plane. We denote these points by M_1 and M_2 , respectively.
- ★ If no base points coincide, but three of them are collinear, then we denote them by M_1, M_2 and M_5 . Due to assumption (iii) we are done.
- ★ If no three base points are collinear, we can label the points arbitrarily. □

Moreover we can choose the moving frame Σ in a way that m_1 equals its origin. The fixed frame Σ_0 is selected in a way that M_1 equals the origin, M_2 is located on the x -axis and the remaining points belong to the xy -plane. Due to Lemma 1 we can assume w.l.o.g. that $A_2 B_3 B_4 (a_3 - a_4) \neq 0$ holds.

The proof of Theorem 1 is based on the following result obtained by Borras et al. [17]: A leg of a given planar pentapod with linear platform is replaced by a leg with platform anchor point $(a, 0, 0)$ and base anchor point $(A, B, 0)$ fulfilling Eq. (6) of [17], which reads as follows under consideration of our special choice of coordinate systems Σ and Σ_0 :

$$[(D_2, D_3, 0) + a(D_4, D_5, D_1)] \begin{pmatrix} A \\ B \\ 1 \end{pmatrix} = 0 \quad (4)$$

with

$$\begin{aligned} D_1 &:= \det(\mathbf{A}, \mathbf{B}, \mathbf{aA}, \mathbf{aB}), & D_2 &:= -\det(\mathbf{a}, \mathbf{B}, \mathbf{aA}, \mathbf{aB}), & D_3 &:= \det(\mathbf{a}, \mathbf{A}, \mathbf{aA}, \mathbf{aB}), \\ D_4 &:= -\det(\mathbf{a}, \mathbf{A}, \mathbf{B}, \mathbf{aB}), & D_5 &:= \det(\mathbf{a}, \mathbf{A}, \mathbf{B}, \mathbf{aA}), \end{aligned} \quad (5)$$

³No three base points can coincide as it yields a contradiction to assumption (iii).

and

$$\mathbf{a} := \begin{pmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}, \quad \mathbf{A} := \begin{pmatrix} A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix}, \quad \mathbf{B} := \begin{pmatrix} 0 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix}, \quad \mathbf{aA} := \begin{pmatrix} a_2 A_2 \\ a_3 A_3 \\ a_4 A_4 \\ a_5 A_5 \end{pmatrix}, \quad \mathbf{aB} := \begin{pmatrix} 0 \\ a_3 B_3 \\ a_4 B_4 \\ a_5 B_5 \end{pmatrix}, \quad (6)$$

then the resulting planar pentapod with linear platform has the same singularity set (and direct kinematics solution) if it is not architecturally singular.

In the following we study Eq. (4) in more detail:

Lemma 2. *A planar pentapod with linear platform fulfilling the assumptions (i,ii,iii) is architecturally singular if and only if $D_1 = D_4 = 0$ holds (with respect to Σ and Σ_0).*

PROOF: Due to Lemma 1 we can assume w.l.o.g. that the following determinant

$$\begin{vmatrix} A_2 & 0 & 0 \\ A_3 & B_3 & a_3 B_3 \\ A_4 & B_4 & a_4 B_4 \end{vmatrix} = A_2 B_3 B_4 (a_3 - a_4) \quad (7)$$

is different from zero. Therefore $D_1 = D_4 = 0$ implies $rk(\mathbf{a}, \mathbf{A}, \mathbf{B}, \mathbf{aA}, \mathbf{aB}) < 5$ which characterizes architecturally singularity (cf. [16, 17, 18]). \square

Based on this lemma we can prove the next one, which reads as follows:

Lemma 3. *For a planar pentapod of \mathcal{P} the condition $D_2 = D_3 = 0$ cannot hold (with respect to Σ and Σ_0).*

PROOF: The proof is done by contradiction as follows: We show that for a planar pentapod with linear platform the condition $D_2 = D_3 = 0$ (with respect to Σ and $\Sigma_0 \Rightarrow A_2 B_3 B_4 (a_3 - a_4) \neq 0$) implies either an architecturally singular design ($\Leftrightarrow D_1 = D_4 = 0$; cf. Lemma 2) or a contradiction to the assumptions (i,ii,iii).

It can easily be seen that $D_2 = 0$ is fulfilled for $a_2 = 0$ ($\Leftrightarrow m_1 = m_2$). Now $D_3 = 0$ simplifies to:

$$-a_3 a_4 a_5 A_2 (A_4 B_5 - A_5 B_4 - A_3 B_5 + A_5 B_3 + A_3 B_4 - A_4 B_3). \quad (8)$$

$a_3 a_4 a_5 = 0$ contradicts assumption (i), A_2 cannot vanish, and the last factor implies the collinearity of M_3, M_4, M_5 , which contradicts assumption (ii). Therefore we can assume for the remaining discussion that $a_2 \neq 0$ holds. We distinguish the following cases:

1. $a_5 \neq 0$: Under this assumption we can solve $D_2 = 0$ for A_5 . Then the numerator of D_3 factors into

$$a_3 a_4 (A_2 B_3 - A_2 B_4 + A_3 B_4 - A_4 B_3) F[20], \quad (9)$$

where the number in the brackets gives the number of terms. As F is also a factor of D_1 and D_4 , the condition $F = 0$ implies an architecturally singular design. Moreover $a_i = 0$ ($\Leftrightarrow m_1 = m_i$) implies the collinearity of M_2, M_j, M_5 for pairwise distinct $i, j \in \{3, 4\}$. Therefore $a_3 a_4 = 0$ contradicts assumption (ii). The vanishing of the third factor of Eq. (9) implies the collinearity of M_2, \dots, M_5 , which contradicts assumption (iii).

2. $a_5 = 0$ ($\Leftrightarrow m_1 = m_5$): Now D_2 and D_3 factors into:

$$-a_2 a_3 a_4 B_5 (A_2 B_3 - A_2 B_4 + A_3 B_4 - A_4 B_3), \quad a_2 a_3 a_4 A_5 (A_2 B_3 - A_2 B_4 + A_3 B_4 - A_4 B_3). \quad (10)$$

$a_2 a_3 a_4 = 0$ contradicts assumption (i) and the last factor implies the collinearity of M_2, M_3, M_4 , which contradicts assumption (ii). Therefore $A_5 = B_5 = 0$ has to hold, but in this case the first and the fifth leg coincide. This closes the proof of Lemma 3. \square

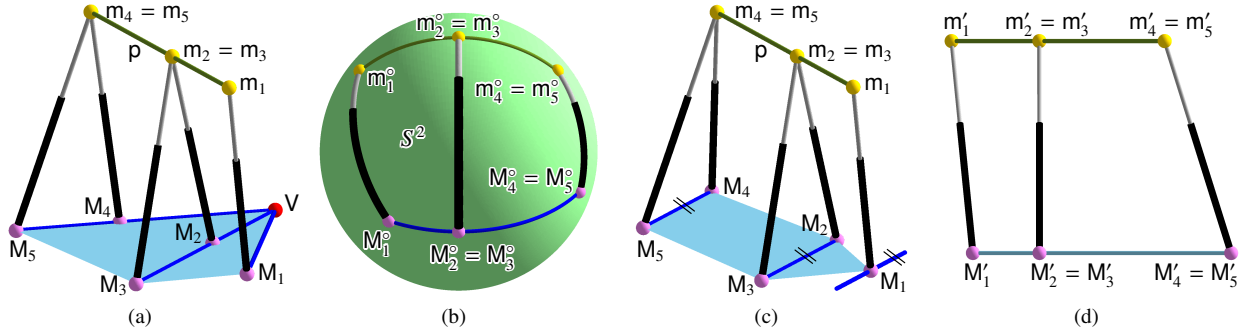


Figure 3: (a) V is a finite point. (b) Spherical image of the pentapod given in (a) with respect to the unit sphere S^2 centered in V . (c) V is an ideal point. (d) Orthogonal projection of the pentapod given in (c) onto a plane orthogonal to V .

Due to this lemma Eq. (4) determines for all planar pentapods of \mathcal{P} a bijection between points of p and lines in the base plane.⁴ Due to the linear relation the lines generate a pencil with vertex V , which can also be an ideal point (\Rightarrow parallel line pencil). According to [19] we are now able to perform a series of leg-replacements in a way that we end up with a non-architecturally singular pentapod of the following type (see Fig. 3a,c):

$$m_2 = m_3, \quad m_4 = m_5, \quad M_2, M_3, V \text{ are collinear}, \quad M_4, M_5, V \text{ are collinear}. \quad (11)$$

Based on this preparatory work we can prove Theorem 1 as follows:

PROOF OF THEOREM 1: We distinguish the following two cases:

1. V is a finite point: In this case the motion of p can only be spherical with center V . Therefore we can also consider the spherical 3-legged manipulator, which we obtain by projecting the pentapod onto the unit sphere S^2 centered in V . Therefore the spherical 3-legged manipulator has to have a self-motion.

Note that the projected base points M_i° as well as the projected platform points m_i° are located on great circles of S^2 . It is well known (cf. [20, Lemma 2] and [21, Theorem 5]) that the 3-legged spherical manipulators illustrated in (see Fig. 3b) can only have self-motions if two platform or base anchor points coincide. Now this is only possible if the line p contains V . In this case the platform of the spherical manipulator collapse into a point and we only get the trivial rotation about the line p as uncontrollable motion while p itself remains fix. Therefore this case does not yield a solution.

2. V is an ideal point: Now the motion of p can only be a planar one orthogonal to the direction of V . Therefore the corresponding planar 3-legged manipulator, which is obtained by an orthogonal projection of the pentapod onto a plane orthogonal to V , also has to have a self-motion. Note that the projected base points M'_i as well as the projected platform points m'_i are collinear (see Fig. 3d). According to [20, 21] this planar 3-legged manipulator can only have a self-motion in one of the following two cases:

- (a) Two platform or base anchor points coincide: This is only possible if the line p contains V . Analogous considerations as in the spherical case show that we do not get a solution.
- (b) The platform and the base are congruent and all legs have equal lengths: In this case the planar 3-legged manipulator has a circular translation. This already implies the solution given in Theorem 1. \square

Remark 1. Note that the case given in Theorem 1 is also not covered by the more than 100 year old results of Darboux, Mannheim, Duporcq and Bricard, respectively, even though no platform anchor points have to coincide. Therefore our study reveals a further lost case beside design (γ) .

⁴For $D_2 = D_3 = 0$ Eq. (4) would factor into $a(D_4A + D_5B + D_1) = 0$, which does not imply such a bijection.

But this case is not novel, as it is already contained within the more general characterization given in Theorem 2⁵ of [21], which reads as follows: *A pentapod has a translational self-motion if and only if the platform can be rotated about the center $\mathbf{m}_1 = \mathbf{M}_1$ into a pose, where the vectors $\overrightarrow{\mathbf{M}_i\mathbf{m}_i}$ for $i = 2, \dots, 5$ fulfill the condition $rk(\overrightarrow{\mathbf{M}_2\mathbf{m}_2}, \dots, \overrightarrow{\mathbf{M}_5\mathbf{m}_5}) \leq 1$.* Therefore a pentapod with a linear platform and a translational self-motion has to have a planar base. \diamond

3. Types of non-planar pentapods

Based on the idea of singular-invariant leg-replacements, which can also be extended to the non-planar case (cf. [2]), one can distinguish different types introduced in this section.

Lemma 4. *The anchor points of a non-planar pentapod with a linear platform can always be relabeled in a way that the following conditions hold:*

1. $\mathbf{M}_1, \dots, \mathbf{M}_4$ span a tetrahedron.
2. If $\mathbf{M}_1, \dots, \mathbf{M}_5$ are pairwise distinct, then $\mathbf{M}_1, \mathbf{M}_i, \mathbf{M}_5$ are not collinear for all $i \in \{2, 3, 4\}$.
If two base anchor points coincide, then \mathbf{M}_1 is none of them.

As the proof is trivial it is left to the reader and we proceed with the coordinatization used within this section: Again we choose the origin of the moving system Σ in \mathbf{m}_1 . The fixed frame Σ_0 is selected in a way that \mathbf{M}_1 equals the origin, \mathbf{M}_2 belongs to the x -axis and \mathbf{M}_3 is located in the xy -plane. All in all this yields $A_2B_3C_4 \neq 0$ and $\mathbf{M}_5 \neq \mathbf{o}$ (zero vector).

Moreover we introduce the following notation: D_{ijk} denotes the determinant of the 4×7 matrix

$$(\mathbf{a}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{aA}, \mathbf{aB}, \mathbf{aC}) \quad \text{with} \quad \mathbf{C} = (0, 0, C_4, C_5)^T, \quad \mathbf{aC} = (0, 0, a_4C_4, a_5C_5)^T \quad (12)$$

after removing the i -th, j -th and k -th column. Now we can state the following lemma:

Lemma 5. *For a non-planar pentapod of \mathcal{P} the condition $D_{167} = D_{157} = D_{156} = 0$ cannot hold (with respect to Σ and Σ_0).*

PROOF: As $\mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$ are linearly independent there exist a unique triple $(\lambda_2, \lambda_3, \lambda_4)$ with $\lambda_2\mathbf{M}_2 + \lambda_3\mathbf{M}_3 + \lambda_4\mathbf{M}_4 = \mathbf{M}_5$. Due to $\mathbf{M}_5 \neq \mathbf{o}$ we have $(\lambda_2, \lambda_3, \lambda_4) \neq (0, 0, 0)$. Now it can easily be seen that the following equivalences hold:

$$D_{167} = 0 \Leftrightarrow \sum_{i=2}^4 \lambda_i a_i A_i = a_5 A_5, \quad D_{157} = 0 \Leftrightarrow \sum_{i=2}^4 \lambda_i a_i B_i = a_5 B_5, \quad D_{156} = 0 \Leftrightarrow \sum_{i=2}^4 \lambda_i a_i C_i = a_5 C_5. \quad (13)$$

Therefore $D_{167} = D_{157} = D_{156} = 0$ implies $\lambda_2 a_2 \mathbf{M}_2 + \lambda_3 a_3 \mathbf{M}_3 + \lambda_4 a_4 \mathbf{M}_4 = a_5 \mathbf{M}_5$. As a_5 cannot equal zero⁶ we can divide both sides by a_5 , which shows that the following relation has to hold:

$$(\lambda_2, \lambda_3, \lambda_4) = \left(\frac{a_2}{a_5} \lambda_2, \frac{a_3}{a_5} \lambda_3, \frac{a_4}{a_5} \lambda_4 \right). \quad (14)$$

In order to get no contradiction with assumption (i) and $\mathbf{M}_5 \neq \mathbf{o}$ the implied three equations only have the following solution: $a_i = a_5$ and $\lambda_j = \lambda_k = 0$ with pairwise distinct $i, j, k \in \{2, 3, 4\}$. For $\lambda_i \neq 1$ the points $\mathbf{M}_1, \mathbf{M}_i, \mathbf{M}_5$ are collinear, which contradicts Lemma 4 and for $\lambda_i = 1$ the i -th leg and the fifth leg coincide; a contradiction. \square

In the following we distinguish two cases with respect to the criterion whether D_{567} vanishes or not. This subdivision was also used by Bricard in [8, Items 12 and 13 of Chapter III] as $D_{567} = 0$ is equivalent to the following geometric condition, which we call the affine relation (AR):

(AR) There exists a singular affinity κ with $\mathbf{M}_i \mapsto \mathbf{m}_i$ for $i = 1, \dots, 5$.

⁵This theorem is originally stated for hexapods but it also holds for pentapods, as its proof is also valid for 5-legged manipulators.

⁶For $a_5 = 0$ we get $\mathbf{m}_1 = \mathbf{m}_5$ and therefore $a_2 a_3 a_4 \neq 0$ has to hold, as otherwise we get a contradiction to assumption (i).

3.1. $D_{567} \neq 0$

Under this assumption we can use the following result of Borras and Thomas [2]: *A leg of a given non-planar pentapod with linear platform is replaced by a leg with platform anchor point $(a, 0, 0)$ and base anchor point (A, B, C) fulfilling Eq. (7) of [2], which reads as follows within our notation:*

$$\begin{pmatrix} D_{267} - aD_{567} & -D_{367} & D_{467} \\ D_{257} & -D_{357} - aD_{567} & D_{457} \\ D_{256} & -D_{356} & D_{456} - aD_{567} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = a \begin{pmatrix} D_{167} \\ D_{157} \\ D_{156} \end{pmatrix} \quad (15)$$

then the resulting pentapod has the same singularity set (and direct kinematics solution) if it is not architecturally singular.

Solving Eq. (15) with Crammer's rule yields:

$$A = \frac{d_1(a)}{d_0(a)}, \quad B = \frac{d_2(a)}{d_0(a)}, \quad C = \frac{d_3(a)}{d_0(a)}. \quad (16)$$

Due to the assumption $D_{567} \neq 0$, the polynomial d_0 is cubic in the unknown a . The other polynomials d_1, d_2, d_3 are of degree 3 or less in a , but due to Lemma 5 one of them has to be cubic, which shows the following result:

Corollary 2. *The locus of base anchor points of singular-invariant leg-replacements of a non-planar pentapod of \mathcal{P} , which does not fulfill the affine relation (AR), is a cubic space curve.*

According to Borras and Thomas [2] we can distinguish different types with respect to the number of roots of $d_0 = 0$, for which the system Eq. (15) is consistent. This yields the following classification:

Theorem 2. *A non-planar pentapod of \mathcal{P} , which does not fulfill the affine relation (AR), belongs to one of the following four types: The cubic of Corollary 2:*

Type 1 is irreducible: There is a bijection σ between \mathbf{p} and this space curve \mathbf{s} (see Fig. 4a).

Type 2 splits up into an irreducible conic \mathbf{q} , located in the finite plane ε and a finite line $\mathbf{g}_1 \notin \varepsilon$, which intersects \mathbf{q} in the point \mathbf{Q} : There is a bijection σ between $\mathbf{p} \setminus \{\mathbf{P}_1\}$ and $\mathbf{q} \setminus \{\mathbf{Q}\}$. Moreover the finite point \mathbf{P}_1 is mapped to \mathbf{g}_1 (see Fig. 4b).

Type 3 splits up into the finite lines \mathbf{l} and the finite skew lines $\mathbf{g}_1, \mathbf{g}_2$, which intersects \mathbf{l} in the point \mathbf{L}_1 and \mathbf{L}_2 , respectively: There is a bijection σ between $\mathbf{p} \setminus \{\mathbf{P}_1, \mathbf{P}_2\}$ and $\mathbf{l} \setminus \{\mathbf{L}_1, \mathbf{L}_2\}$. Moreover the finite point \mathbf{P}_i is mapped to \mathbf{g}_i for $i = 1, 2$ (see Fig. 4c).

Type 4 splits up into the finite lines $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$, which are not coplanar but intersect each other in the finite point \mathbf{V} : All points of $\mathbf{p} \setminus \{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ are mapped to \mathbf{V} . Moreover the finite point \mathbf{P}_i is mapped to \mathbf{g}_i for $i = 1, 2, 3$ (see Fig. 4d).

Note that in Type i we have $4 - i$ points $\mathbf{W}_1\sigma^{-1}, \dots, \mathbf{W}_{4-i}\sigma^{-1}$ on \mathbf{p} (counted with algebraic multiplicity) for $i = 1, \dots, 4$, which are mapped by σ to ideal points $\mathbf{W}_1, \dots, \mathbf{W}_{4-i}$ of the base. These points of \mathbf{p} have the special property that their trajectory is in a plane orthogonal to the respective ideal point. Therefore each of these point pairs determines a so-called Darboux condition (cf. [8, Item 6 of Chapter II] and [22, Section 4.1]). Any other finite point of \mathbf{p} determines a so-called sphere condition; i.e. it is located on a sphere centered in the corresponding finite base anchor point.

The points \mathbf{P}_i have the special property of possessing circular trajectories, i.e. their path is planar and spherical at the same time.

Note that the ideal point \mathbf{U} of \mathbf{p} is in all four cases mapped by σ onto a finite point $\mathbf{U}\sigma$. Therefore this point pair determines a so-called Mannheim condition, which is the inverse of the Darboux condition; i.e. a plane of the moving system orthogonal to \mathbf{p} slides through a finite point of the base.

Remark 2. The above given correspondence between points on \mathbf{p} and points on the base can also be seen as the correspondence of point paths of \mathbf{p} and the centers of their osculating spheres. It is an old result of Schönflies [23] that this correspondence is cubic. \diamond

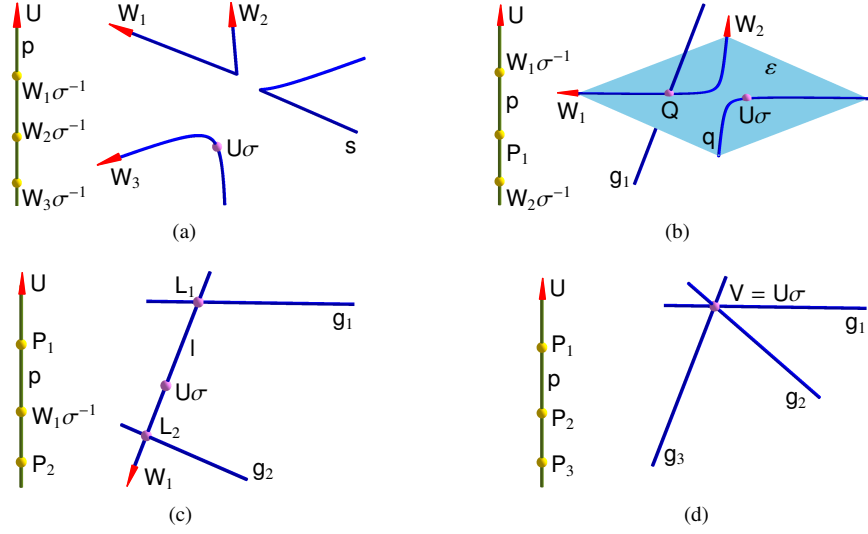


Figure 4: Sketch of the types of non-planar pentapods of \mathcal{P} , which do not fulfill the affine relation: (a) Type 1. (b) Type 2. (c) Type 3. (d) Type 4. Pentapods fulfilling the affine relation belong to Type 5.

3.2. $D_{567} = 0$

Theorem 3. *For a non-planar pentapod of \mathcal{P} , which fulfills the affine relation (AR), the ideal point U of p is related to an ideal element of the base within the correspondence implied by singular-invariant leg-replacements. Pentapods with this property belong to Type 5.*

PROOF: Due to Lemma 5 not all three determinants D_{156} , D_{157} , D_{167} can be equal to zero. Therefore we can assume e.g. that $D_{167} \neq 0$ ⁷ holds. Then similar considerations to those given in [2] yield the following correspondence:

$$\begin{pmatrix} D_{267} & -D_{367} & D_{467} \\ D_{126} - aD_{156} & -D_{136} & D_{146} + aD_{167} \\ D_{127} - aD_{157} & aD_{167} - D_{137} & D_{147} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = a \begin{pmatrix} D_{167} \\ 0 \\ 0 \end{pmatrix}. \quad (17)$$

Introducing homogeneous coordinates to the first of these three equations show that the ideal point U of p is mapped to an ideal element (ideal point, ideal line or the complete ideal plane) of the base. \square

Lemma 6. *If a pentapod of Type 5 has a self-motion, then the ideal element has to be a point W . Moreover the self-motion is a Schönflies motion with axis direction W .*

PROOF: We prove by contradiction that U cannot be mapped to more than one ideal point W of the base: Assume that U can be connected with two distinct ideal points W_1 and W_2 of the base. Then these special two "legs" correspond to two angle conditions (cf. [8, Item 7 of Chapter II] and [22, Section 4.1]); i.e. the angle enclosed by U and W_i ($i = 1, 2$) has to be constant during the self-motion. As this already fixes the orientation of p , the pentapod can only have a translational self-motion, which implies the coplanarity of the base (cf. Remark 1); a contradiction.

As the angle enclosed by U and W has to be constant, the self-motion of p can only be a Schönflies motion with axis direction W . \square

In the following we analyze the Types 1-5 separately with respect to the existence of self-motions. Type 4 can be discussed similar to the planar case, as singular-invariant leg-replacements can be used to get the same pentapod

⁷For $D_{156} \neq 0$ or $D_{157} \neq 0$ the argumentation can be done analogously.

illustrated in Fig. 3a with the sole difference that the three lines through the finite point V are not coplanar. Under consideration of [21, Theorem 6] an analogous argumentation as in item 1 of the proof of Theorem 1 can be done for Type 4, which shows the following result:

Theorem 4. *A pentapod of Type 4 cannot possess a self-motion.*

For the discussion of the remaining four types we apply the theory of bonds for pentapods with linear platform, which is the content of the next section.

4. Bond theory for pentapods with linear platform

Readers, who are not familiar with the following terms of algebraic geometry (ideal, variety, Gröbner base, radical), are recommended to look up these basics in the articles [24] and [25], before studying this section.

It was shown by Husty [26] that the sphere condition equals a homogeneous quadratic equations in the Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$. For our choice of the moving frame Σ the sphere condition Λ_i simplifies to:

$$\begin{aligned} \Lambda_i : & (a_i^2 + A_i^2 + B_i^2 + C_i^2 - R_i^2)(e_0^2 + e_1^2 + e_2^2 + e_3^2) - 2a_iA_i(e_0^2 + e_1^2 - e_2^2 - e_3^2) - 4a_iB_i(e_0e_3 + e_1e_2) \\ & + 4a_iC_i(e_0e_2 - e_1e_3) - 4A_i(e_0f_1 - e_1f_0 - e_2f_3 + e_3f_2) + 4A_i(e_0f_1 - e_1f_0 + e_2f_3 - e_3f_2) \\ & + 4B_i(e_0f_2 - e_1f_3 - e_2f_0 + e_3f_1) + 4C_i(e_0f_3 + e_1f_2 - e_2f_1 - e_3f_0) + 4(f_0^2 + f_1^2 + f_2^2 + f_3^2) = 0, \end{aligned} \quad (18)$$

where R_i denotes the radius of the sphere centered in M_i on which m_i is located.

Now, all real points of the Study parameter space P^7 (7-dimensional projective space), which are located on the so-called Study quadric $\Psi : \sum_{i=0}^3 e_i f_i = 0$, correspond to an Euclidean displacement, with exception of the 3-dimensional subspace $e_0 = e_1 = e_2 = e_3 = 0$, as its points cannot fulfill the condition $e_0^2 + e_1^2 + e_2^2 + e_3^2 \neq 0$. The translation vector $\mathbf{t} := (t_1, t_2, t_3)^T$ and the rotation matrix \mathbf{R} of the corresponding Euclidean displacement $\mathbf{m}_i \mapsto \mathbf{R}\mathbf{m}_i + \mathbf{t}$ are given by:

$$t_1 = -2(e_0f_1 - e_1f_0 + e_2f_3 - e_3f_2), \quad t_2 = -2(e_0f_2 - e_2f_0 + e_3f_1 - e_1f_3), \quad t_3 = -2(e_0f_3 - e_3f_0 + e_1f_2 - e_2f_1),$$

and

$$\mathbf{R} = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\ 2(e_1e_2 + e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 - e_0e_1) \\ 2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}, \quad (19)$$

if the normalizing condition $e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$ is fulfilled.

Now the solution for the direct kinematics of a pentapod with linear platform can be written as the algebraic variety of the ideal spanned by $\Psi, \Lambda_1, \dots, \Lambda_5, e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$. In the general case the dimension of this variety equals 1, as we also obtain the 1-dimensional rotation about the line \mathbf{p} . In the following we present another kinematic mapping, where we get rid of this redundancy.

4.1. Kinematic mapping for pentapods with linear platform

With respect to the following nine homogeneous motion parameters $(n_0 : x_0 : \dots : x_3 : y_0 : \dots : y_3)$ with $n_0 := f_0^2 + f_1^2 + f_2^2 + f_3^2$ and

$$\begin{aligned} x_0 &:= 2(e_0^2 + e_1^2 + e_2^2 + e_3^2), & y_0 &:= 4(-e_0f_1 + e_1f_0 + e_2f_3 - e_3f_2), \\ x_1 &:= 2(-e_0^2 - e_1^2 + e_2^2 + e_3^2), & y_1 &:= 4(e_0f_1 - e_1f_0 + e_2f_3 - e_3f_2), \\ x_2 &:= -4(e_0e_3 + e_1e_2), & y_2 &:= 4(e_0f_2 - e_1f_3 - e_2f_0 + e_3f_1), \\ x_3 &:= 4(e_0e_2 - e_1e_3) & y_3 &:= 4(e_0f_3 + e_1f_2 - e_2f_1 - e_3f_0), \end{aligned}$$

the sphere condition of Eq. (18) is linear; i.e.

$$\Lambda_i : \quad \frac{1}{2}(a_i^2 + A_i^2 + B_i^2 + C_i^2 - R_i^2)x_0 + a_iA_ix_1 + a_iB_ix_2 + a_iC_ix_3 + a_iy_0 + A_iy_1 + B_iy_2 + C_iy_3 + 4n_0 = 0. \quad (20)$$

For $x_0 = 1$ the Euclidean displacement of \mathbf{m}_i is given by:

$$\begin{pmatrix} a_i \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -a_i x_1 - y_1 \\ -a_i x_2 - y_2 \\ -a_i x_3 - y_3 \end{pmatrix}. \quad (21)$$

Note that $(n_0 : x_0 : \dots : x_3 : y_0 : \dots : y_3)$ can be interpreted as a point in the 8-dimensional projective space P^8 . In order to determine the image of the kinematic mapping, we compute a Gröbner bases of the ideal generated by $n_0 - (f_0^2 + f_1^2 + f_2^2 + f_3^2), \dots, y_3 - 4(e_0 f_3 + e_1 f_2 - e_2 f_1 - e_3 f_0)$ and $e_0 f_0 + e_1 f_1 + e_2 f_2 + e_3 f_3$ eliminating the Study parameters e_0, \dots, f_3 . It contains three quadric elements in the remaining variables, namely:

$$\Phi_1 : x_1^2 + x_2^2 + x_3^2 - x_0^2 = 0, \quad \Phi_2 : y_1^2 + y_2^2 + y_3^2 - 8x_0 n_0 = 0, \quad \Phi_3 : x_1 y_1 + x_2 y_2 + x_3 y_3 - x_0 y_0 = 0. \quad (22)$$

Because the degree of the image variety (which is computed from the Gröbner bases) is equal to 8, the three quadrics generate the ideal of the image variety \mathcal{I} . The image itself consists of all real points in the zero set with $x_0 \neq 0$. Note that \mathcal{I} is of dimension 5 instead of 6, as we removed the rotations around the line \mathbf{p} , which do not change the spherical condition.

Definition 2. *The intersection of \mathcal{I} with the five hyperplanes of Eq. (20) is the complex configuration set \mathcal{C} of the pentapod. Its real points are called real configurations.*

Lemma 7. *A generic pentapod with linear platform has eight solutions for the direct kinematics over \mathbb{C} .*

PROOF: For the solution of the direct kinematics we have to intersect the image variety \mathcal{I} with the five hyperplanes of P^8 given by Eq. (20). In the generic case the five hyperplanes have a linear 3-space L in common, which intersects \mathcal{I} in a finite number of points (= complex configuration set \mathcal{C}) whose cardinality equals the degree of \mathcal{I} . \square

Remark 3. Lemma 7 fits with the results obtained in [27], where the number of eight solutions for the direct kinematics problem over \mathbb{C} was given for planar pentapods with linear platform. Due to Lemma 7 this number also holds for the non-planar case. \diamond

4.2. Bonds

Let us assume that the pentapod under consideration has a 1-dimensional self-motion. In this case the complex configuration set \mathcal{C} of Definition 2 is a curve. The points on this configuration curve, which do not correspond to Euclidean displacements as they fulfill $x_0 = 0$, are the so-called bonds. This gives a rough idea of bonds, which is developed in detail within this section.

The intersection of the 5-fold \mathcal{I} with the hyperplane $x_0 = 0$ yields the so-called boundary of \mathcal{I} . A Gröbner bases computation of the ideal generated by the generators of the ideal of \mathcal{I} and by x_0 reveals that some perfect squares, for instance $(x_2 y_3 - x_3 y_2)^2$, are contained in this ideal. Therefore the radical contains the elements $\Gamma_4, \Gamma_5, \Gamma_6$ below. Its zero set is a 4-fold of degree 4, hence a variety of minimal degree (degree = codimension + 1). It is given by the following set of equations:

$$\Gamma_1 : x_1^2 + x_2^2 + x_3^2 = 0, \quad \Gamma_2 : y_1^2 + y_2^2 + y_3^2 = 0, \quad \Gamma_3 : x_1 y_1 + x_2 y_2 + x_3 y_3 = 0, \quad (23)$$

$$\Gamma_4 : x_1 y_2 - x_2 y_1 = 0, \quad \Gamma_5 : x_1 y_3 - x_3 y_1 = 0, \quad \Gamma_6 : x_2 y_3 - x_3 y_2 = 0, \quad (24)$$

and therefore it is independent of y_0 and n_0 .

Definition 3. *The intersection of \mathcal{C} with the boundary of \mathcal{I} is the set \mathcal{B} of bonds.*

Due to $x_0 = 0$ the set \mathcal{B} of bonds is independent of the leg lengths R_i (cf. Eq. (20)), and therefore \mathcal{B} only depends on the geometry of the pentapod with linear platform.

Remark 4. Bonds of pentapods were already introduced in [21] and [28]. In [21] they were defined with respect to the Study parameters and in [28] with respect to a special compactification of $SE(3)$, which can be seen as a generalization of the method presented above. But both approaches are not suited for the study of pentapods with linear platform, due to the rotational redundancy about \mathbf{p} . \diamond

Type	Necessary Condition 1	Necessary Condition 2	Sufficiency & Leg Parameters
1	Theorem 6	Theorem 9	Theorem 13
2	Theorem 7	Theorem 10	Theorem 14
3	Theorem 8		
5	Theorem 5	Theorem 11	Theorem 12

Table 1: Overview of the Theorems, where the first and second necessary condition for mobility arising from bond theory are applied to the remaining types (for Type 4 see Theorem 4) of non-planar pentapods of \mathcal{P} . Moreover the theorems are listed, where the sufficiency of the obtained necessary conditions is proven. Within these theorems also the leg parameters for a self-motion are given.

In the following we state two necessary conditions for the existence of self-motions in terms of bonds:

Assume that a pentapod with linear platform has a 1-dimensional configuration set \mathcal{C} ; i.e. there exists a configuration curve \mathbf{c} on the 5-fold \mathcal{I} . Therefore the corresponding bond set \mathcal{B} contains at least one bond β (up to conjugation). Therefore the existence of a bond is the first necessary condition.

As we have mobility 1, the pentapod with linear platform fulfills the necessary condition of being infinitesimal movable in each pose of the 1-dimensional motion of \mathbf{p} . This is equivalent with the existence of a 1-dimensional tangent space in each point of the configuration curve \mathbf{c} . As $\beta \in \mathbf{c}$ holds, this implies a second necessary condition for mobility 1.

Remark 5. Therefore the 3-space L has to intersect the 5-fold \mathcal{I} in the bond β at least of multiplicity 2. As this also holds for the conjugate of β , only one pair of conjugate complex bonds can exist (due to Bezout's theorem). Moreover if β (and its conjugate) is a singular point of \mathcal{I} then the second necessary condition is trivially fulfilled. \diamond

5. Non-planar pentapods of \mathcal{P} with self-motions

The following determination of all non-planar pentapods of \mathcal{P} with self-motions is based on the two necessary conditions given in Section 4.2. For an overview of the workflow/results structured by types please see Table 1.

5.1. First necessary condition

In this section we only apply the first necessary condition, namely the existence of a bond.

Theorem 5. *The base anchor points of possible leg-replacements of a pentapod of Type 5 with self-motions have to be located on an irreducible cubic ellipse \mathcal{S}^* on a cylinder of revolution.*

PROOF: Given is a pentapod of Type 5. W.l.o.g. we can assume that M_1, \dots, M_4 span a tetrahedron. Due to the properties of Type 5 (cf. Lemma 6) we can replace the sphere condition implied by the fifth leg by an angle condition; i.e. the angle φ enclosed by the ideal point m_5 of \mathbf{p} and the ideal point M_5 of the base is constant.

Now we can choose the fixed frame Σ_0 that M_1 is the origin, M_5 is the ideal point of the x -axis, and M_2 is located in the xy -plane. The moving frame Σ is chosen in a way that m_1 is its origin. With respect to these coordinate systems the angle condition reads as follows:

$$\angle_5 : x_1 + wx_0 = 0 \quad (25)$$

where w denotes $\arccos(\varphi)$. The equations of Λ_i for $i = 1, 2, 3, 4$ are given in Eq. (20) under consideration of $A_1 = B_1 = C_1 = C_2 = a_1 = 0$.

We set $x_0 = 0$ and start the computation of the bonds: As M_1, \dots, M_4 are non-planar ($\Leftrightarrow K \neq 0$ with

$$K = A_2(B_3C_4 - B_4C_3) + B_2(A_4C_3 - A_3C_4) \quad (26)$$

holds) we can solve $\Lambda_1, \dots, \Lambda_4, \angle_5$ for n_0, x_1, y_1, y_2, y_3 w.l.o.g.. As we get $x_1 = 0$, the condition Γ_1 implies $x_2 = \pm ix_3$. W.l.o.g. we only discuss the upper branch as the lower one can be done analogously.

Now the numerator of Γ_3 factors into $x_3F_3[26]$ and the numerator of Γ_4 splits up into $x_3F_4[10]$. Therefore we have to distinguish two cases:

1. $x_3 = 0$: Now the numerator of Γ_2 factors into $y_0^2 F_2[41]$, where F_2 is quadratic with respect to a_4 . The discriminant of F_2 with respect to a_4 equals:

$$-K^2 \left[(a_2 A_3 - a_3 A_2)^2 + (a_2 B_3 - a_3 B_2)^2 + a_2^2 C_3^2 \right]. \quad (27)$$

This expression can never be greater than zero. It is equal to zero if the following three conditions are fulfilled:

$$a_2 A_3 - a_3 A_2 = 0, \quad a_2 B_3 - a_3 B_2 = 0, \quad a_2 C_3 = 0. \quad (28)$$

As $a_2 = a_3 = 0$ contradicts assumption (i) we have to discuss the following cases:

- (a) $a_2 = 0$: This implies $A_2 = B_2 = 0$ and therefore the first and second leg coincide; a contradiction.
- (b) $a_3 = 0, a_2 \neq 0$: We get $A_3 = B_3 = C_3 = 0$ and therefore the first and third leg coincide; a contradiction.
- (c) $a_2 a_3 \neq 0$: We get $C_3 = 0, A_3 = a_3 A_2 / a_2$ and $B_3 = a_3 B_2 / a_2$ but this implies $K = 0$; a contradiction.

As a consequence the case $x_3 = 0$ does not imply any solution to our problem.

2. $x_3 \neq 0$: In this case $F_3[26] = 0$ and $F_4[10] = 0$ have to hold. Computing the resultant of these two expressions with respect to y_0 yields $-x_3 K(G_r + iG_c)$ with

$$\begin{aligned} G_r &= [a_4 B_3(a_2 - a_3) + a_3 B_4(a_4 - a_2)] B_2 + a_2(a_3 - a_4)(B_3 B_4 - C_3 C_4), \\ G_c &= -[a_4 C_3(a_2 - a_3) + a_3 C_4(a_4 - a_2)] B_2 - a_2(a_3 - a_4)(B_3 C_4 + B_4 C_3). \end{aligned} \quad (29)$$

These two expressions do not depend on the A_i coordinates. We denote the orthogonal projection of the base anchor point M_i onto the yz -plane of the fixed frame Σ_0 by M'_i for $i = 1, \dots, 4$. In the following we want to show that M'_1, \dots, M'_4 has to be pairwise distinct as well as m_1, \dots, m_4 . The proof is done by contradiction, where the following cases have to be discussed:

- (a) Two platform anchor points coincide: W.l.o.g. we can set $m_1 = m_2$; i.e. $a_2 = 0$. Then G_r and G_c simplify to

$$G_r = -a_3 a_4 B_2 (B_3 - B_4), \quad G_c = a_3 a_4 B_2 (C_3 - C_4). \quad (30)$$

As $a_3 a_4 = 0$ contradicts assumption (i), and $M'_3 = M'_4$ ($\Rightarrow M_3, M_4, M_5$ collinear) implies a contradiction to assumption (ii) we remain with the case $M'_1 = M'_2$:

In this case $m_1 = m_2$ is located on a circle in a plane orthogonal to the axis of the Schönflies motion. As the point $m_1 = m_2$ (and therefore the complete line p) cannot be translated in direction of M_5 , the problem reduces to a planar one (projection to the yz -plane of Σ_0). As a circular translation of the resulting planar manipulator is not possible (as otherwise the base has to be planar; cf. Remark 1), anchor points have to coincide (cf. item 2 in the proof of Theorem 1).

- i. If further base anchor points coincide beside $M'_1 = M'_2$ we get again a contradiction to the assumption M_1, \dots, M_4 are non-planar.
 - ii. Further platform anchor points (beside $m'_1 = m'_2$) can only coincide without contradicting assumption (i) or (ii) if p is parallel to the axis of the Schönflies motion. In this case the platform of the planar manipulator collapse into a point and we only get the trivial rotation about the line p as uncontrollable motion while p itself remains fix.
- (b) No platform anchor points coincide and two projected base anchor points coincide: W.l.o.g. we can assume that $M'_1 = M'_2$ holds; i.e. $B_2 = 0$. Then G_r and G_c simplify to

$$G_r = a_2(a_3 - a_4)(B_3 B_4 - C_3 C_4), \quad G_c = -a_2(a_3 - a_4)(B_3 C_4 + B_4 C_3), \quad (31)$$

which cannot vanish without contradiction.

As a consequence M'_1, \dots, M'_4 have to be pairwise distinct, as well as m_1, \dots, m_4 . Therefore there exists a uniquely defined Möbius transformation τ with $m_i \mapsto M'_i$ for $i = 1, 2, 3$ (cf. Section 1.1); i.e. $B_i + iC_i = \tau(a_i)$ holds with τ of Eq. (1) and

$$z_1 = \frac{a_2 - a_3}{B_2 - B_3 - iC_3}, \quad z_2 = 0, \quad z_3 = \frac{a_2 B_3 - B_2 a_3 + i a_2 C_3}{(B_2 - B_3 - iC_3)(B_3 + iC_3)B_2}, \quad z_4 = \frac{a_2 a_3 B_2}{B_3 + iC_3}. \quad (32)$$

Now it can easily be verified that the conditions $G_r = 0$ and $G_c = 0$ of Eq. (29) imply that τ also maps m_4 onto M'_4 . As M_1, \dots, M_4 span a tetrahedron, M'_1, \dots, M'_4 has to be located on a circle. Therefore the base anchor points of possible leg-replacements have to belong to the cylinder of revolution Θ through M_1, \dots, M_4 with generators in direction of M_5 .

In the following we study the possible leg-replacements for this case in more detail: By homogenizing the matrix of Eq. (12) it is not difficult to see that the corresponding matrix reads as follows:

$$\begin{pmatrix} a_2 & A_2 & B_2 & 0 & a_2 A_2 & a_2 B_2 & 0 \\ a_3 & A_3 & B_3 & C_3 & a_3 A_3 & a_3 B_3 & a_3 C_3 \\ a_4 & A_4 & B_4 & C_4 & a_4 A_4 & a_4 B_4 & a_4 C_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \quad (33)$$

As M_1, \dots, M_4 are not coplanar, we have $D_{167} \neq 0$. But due to $D_{156} = 0$ and $D_{157} = 0$ Eq. (17) simplifies to

$$\begin{pmatrix} D_{267} & -D_{367} & D_{467} \\ D_{126} & -D_{136} & D_{146} + aD_{167} \\ D_{127} & aD_{167} - D_{137} & D_{147} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = a \begin{pmatrix} D_{167} \\ 0 \\ 0 \end{pmatrix}. \quad (34)$$

Solving this system of linear equations yields a solution of the form given in Eq. (16), but now $d_0(a)$ is a quadratic expression in a and $d_1(a)$ a cubic one. Therefore the base anchor points belong to a cubic curve s^* , which has to be located on Θ .

Moreover s^* cannot split up into three generators as M'_1, \dots, M'_4 are pairwise distinct. The cubic s^* can also not split up into a conic q and a generator g_1 for the following reason: This case equals Type 2 where P_1 is not a finite point but the ideal point U of p . Therefore M_i has to be located on $q \setminus \{Q\}$, as otherwise m_i equals U , which does not yield a sphere condition. As this has to hold for $i = 1, \dots, 4$ we get a contradiction to the non-planarity assumption.

Therefore the cubic curve has to be an irreducible cubic curve s^* on a cylinder of revolution. From the following theorem of projective geometry it is clear that the ideal plane cannot be an osculating plane of s^* or be tangent to it: *The osculating plane in a point X of a cubic equals the tangent plane to the cone of chords with respect X along the tangent of X (which is a generator of the cone of chords).* Therefore the remaining two intersection points of s^* with the plane at infinity are conjugate complex. \square

Theorem 6. *The irreducible cubic s of a pentapod of Type 1 with self-motions has to be a cubic ellipse located on a cylinder of revolution.*

PROOF: s has at least one real intersection point with the ideal plane, which is denoted by M_4 . As the ideal point m_5 of p is mapped to a finite point M_5 of the base, the point $M_4 \sigma^{-1}$ has to be a finite point of p , which is denoted by m_4 . Therefore this point pair (M_4, m_4) determines a Darboux condition Ω_4 and the point pair (M_5, m_5) a Mannheim condition Π_5 . The remaining three point pairs (M_i, m_i) imply sphere conditions Λ_i for $i = 1, 2, 3$.

Moreover we choose the fixed frame Σ_0 that M_1 is the origin, M_4 the ideal point of the x -axis and M_2 is located in the xy -plane. Moreover we can define the moving frame Σ in a way that m_1 is its origin. With respect to these coordinate systems our conditions can be written as:

$$\Omega_4 : p_4 x_0 + a_4 x_1 + y_1 = 0, \quad \Pi_5 : p_5 x_0 + A_5 x_1 + B_5 x_2 + C_5 x_3 + y_0 = 0, \quad (35)$$

where $(p_4, 0, 0)^T$ are the coordinates of the intersection point of the Darboux plane and the x -axis of Σ_0 , and $(p_5, 0, 0)^T$ are the coordinates of the intersection point of the Mannheim plane and the x -axis of Σ . The equations of Λ_i for $i = 1, 2, 3$ are given in Eq. (20) under consideration of $A_1 = B_1 = C_1 = C_2 = a_1 = 0$.

We set $x_0 = 0$ and start the computation of the bonds: Due to the properties of Type 1 no four base points can be coplanar ($\Rightarrow B_2C_3C_5 \neq 0$) and no two platform anchor points can coincide. Under these assumptions we can solve the equations $\Lambda_1, \Lambda_2, \Lambda_3, \Omega_4, \Pi_5$ for y_0, y_1, y_2, y_3, n_0 . Now the numerator of Γ_4 factors into x_1F with

$$F := (a_2A_2 - a_4A_2 - a_2A_5)x_1 + (a_2B_2 - a_4B_2 - a_2B_5)x_2 - a_2C_5x_3. \quad (36)$$

Therefore we have to distinguish two cases:

1. $x_1 \neq 0$: In this case $F = 0$ has to hold, which can be solve w.l.o.g. for x_3 . Then the numerator of Γ_6 factors into x_2G with

$$G := (A_2x_1 + B_2x_2)(a_2 - a_4)(a_3C_3 - a_3C_5 - a_4C_3) + (A_3x_1 + B_3x_2)(a_3 - a_4)a_2C_5 - (A_5x_1 + B_5x_2)(a_3 - a_4)a_2C_3. \quad (37)$$

We distinguish two cases:

- (a) $x_2 \neq 0$: In this case $G = 0$ has to hold. We define $H := A_2(a_2 - a_4)(a_3C_3 - a_3C_5 - a_4C_3) + A_3(a_3 - a_4)a_2C_5 - A_5(a_3 - a_4)a_2C_3$ and discuss the following two cases:

- i. $H \neq 0$: Under this assumption we can solve $G = 0$ for x_1 . Then the remaining equations only imply one condition which is quadratic with respect to B_3 . The discriminant of this condition with respect to B_3 equals:

$$-H^2 \left[(a_2A_2 - a_4A_2 - a_2A_5)^2 + (a_2B_2 - a_4B_2 - a_2B_5)^2 + a_2^2C_5^2 \right]. \quad (38)$$

Therefore B_3 cannot be real; a contradiction.

- ii. $H = 0$: We can solve $H = 0$ for A_3 w.l.o.g.. Then we can solve the $G = 0$ for B_3 w.l.o.g., which already yields the contradiction, as now the points M_1, M_2, M_3, M_5 are coplanar.

- (b) $x_2 = 0$: Now the numerator of Γ_1 factors into

$$x_1^2 \left[A_2^2(a_2 - a_4)^2 - 2a_2A_2A_5(a_2 - a_4) + (A_5^2 + C_5^2)a_2^2 \right]. \quad (39)$$

The discriminant with respect to A_2 equals $-C_5^2$ and therefore we get a contradiction.

2. $x_1 = 0$: From Γ_1 we get $x_2 = \pm x_3i$. In the following we only discuss the case $x_2 = x_3i$, as the other one can be done analogously. Now the numerator of Γ_6 factors into $x_3^2(G_r + iG_c)$ with

$$G_r := (B_2B_3 - B_3B_5 + C_3C_5)a_2 - B_2(B_3 - B_5)a_3, \quad G_c := B_2(C_3 - C_5)a_3 - (B_2C_3 - B_3C_5 - B_5C_3)a_2. \quad (40)$$

This are the corresponding expressions to Eq. (29). Therefore the two point sets M'_1, M'_2, M'_3, M'_5 and m_1, m_2, m_3, m_5 are again Möbius equivalent. As a consequence \mathbf{s} has to be located on the cylinder of revolution Θ through M'_1, M'_2, M'_3, M'_5 with generators in direction of M_4 . By using again the theorem of projective geometry given at the end of the last proof we are done. \square

Theorem 7. *The conic \mathbf{q} of a pentapod of Type 2 with self-motions has to be located on a cylinder of revolution, where one generator is the line \mathbf{g}_1 .*

PROOF: The proof can be done analogously to the one of Theorem 6. In order to streamline the presentation it is given in Appendix A. It should only be noted that \mathbf{q} can be an ellipse or a circle, respectively, as \mathbf{q} is the planar section of a cylinder of revolution. \square

Theorem 8. *A pentapod of Type 3 cannot possess a self-motion.*

PROOF: In order to improve the readability of the paper the proof of the non-existence of pentapods of Type 3 with self-motions is given in Appendix B. \square

5.2. Second necessary condition

Due to the obtained results only pentapods of Type 1,2,5 remain as candidates for self-motions. In this section we check them with respect to the second necessary condition.

Theorem 9. *The irreducible cubic \mathbf{s} of a pentapod of Type 1 with self-motions has to be a straight cubic circle.*

PROOF: Due to Theorem 6 the cubic \mathbf{s} has three pairwise distinct points at infinity, which are denoted by M_2, M_3, M_4 . Note that M_4 is real and that M_2, M_3 are conjugate complex; i.e. $\overline{M_2} = M_3$. The corresponding platform anchor points are denoted by m_2, m_3, m_4 where $\overline{m_2} = m_3$ holds. Therefore we get three Darboux conditions Ω_i implied by the point pairs (M_i, m_i) for $i = 2, 3, 4$. The ideal point of the line \mathbf{p} is again denoted by m_5 and its corresponding base anchor point with M_5 . Therefore this point pair implies one Mannheim condition Π_5 . The pentapod is completed by a sphere condition Λ_1 determined by the two finite points M_1 and m_1 .

The fixed frame Σ_0 is chosen that M_1 is the origin and that M_2 and M_3 are located in the xy -plane in direction $(1, B_2, 0)$ and $(1, \overline{B_2}, 0)$, respectively. As M_2, M_3, M_4 cannot be collinear, M_4 is the ideal point in direction of $(A_4, B_4, 1)$. Moreover we locate the origin of the moving frame Σ in m_1 . With respect to these coordinate systems Σ and Σ_0 our conditions can be written as:

$$\begin{aligned}\Omega_j : \quad & p_j x_0 + a_j x_1 + a_j \overline{B_j} x_2 + y_1 + \overline{B_j} y_2 = 0, \\ \Omega_4 : \quad & p_4 x_0 + a_4 A_4 x_1 + a_4 B_4 x_2 + a_4 x_3 + A_4 y_1 + B_4 y_2 + y_3 = 0, \\ \Pi_5 : \quad & p_5 x_0 + A_5 x_1 + B_5 x_2 + C_5 x_3 + y_0 = 0,\end{aligned}\tag{41}$$

where $(p_j, 0, 0)^T$ for $j = 2, 3$ are the coordinates of the intersection point of the Darboux plane and the x -axis of Σ_0 , $(0, 0, p_4)^T$ are the coordinates of the intersection point of the Darboux plane and the z -axis of Σ_0 , and $(p_5, 0, 0)^T$ are the coordinates of the intersection point of the Mannheim plane and the x -axis of Σ . The equation of Λ_1 is given in Eq. (20) under consideration of $A_1 = B_1 = C_1 = a_1 = 0$.

With respect to the chosen frames Σ and Σ_0 the first necessary condition is fulfilled if

$$B_r = \frac{A_4 B_4}{A_4^2 + 1}, \quad B_c = \pm \frac{\sqrt{A_4^2 + B_4^2 + 1}}{A_4^2 + 1},\tag{42}$$

holds with $B_2 = B_r + iB_c$ and $B_c \neq 0$. Then the bond β reads as follows:

$$n_0 = 0, \quad x_0 = 0, \quad x_1 = -\overline{B_2}, \quad x_2 = 1, \quad x_3 = A_4 \overline{B_2} - B_4,\tag{43}$$

$$y_0 = \overline{B_2}(A_5 - A_4 C_5) - B_5 + B_4 C_5, \quad y_1 = \overline{B_2} \overline{a_2}, \quad y_2 = -\overline{a_2}, \quad y_3 = -\overline{a_2}(A_4 \overline{B_2} - B_4).\tag{44}$$

Now we apply the second necessary condition; i.e. the eight tangent-hyperplanes to $\Phi_1, \Phi_2, \Phi_3, \Lambda_1, \Omega_2, \Omega_3, \Omega_4, \Pi_5$ in the bond β have to have a line in common. Therefore we compute the gradients of these eight hypersurfaces with respect to the unknown $n_0, x_0, \dots, x_3, y_0, \dots, y_3$ in the bond β . The resulting 8×9 matrix \mathbf{J} has rank 8 ($\Rightarrow \beta$ is a regular point of the 5-fold \mathcal{I}). For the necessary condition $rk(\mathbf{J}) < 8$ the determinants of all 8×8 submatrices of \mathbf{J} have to vanish. The numerator of the determinant of the 8×8 submatrix of \mathbf{J} , obtained by removing the column stemming from the partial derivative with respect to y_2 , factors into $(ia_r - a_c - ia_4)(ia_r + a_c)(A_4^2 + B_4^2 + 1)L_1 L_2$ with

$$L_1 = A_4 \sqrt{A_4^2 + B_4^2 + 1} \mp iB_4, \quad L_2 = B_5 + B_5 A_4^2 - B_4 C_5 - A_4 A_5 B_4 \pm i(A_5 - A_4 C_5) \sqrt{A_4^2 + B_4^2 + 1},\tag{45}$$

and $a_2 = a_r + ia_c$, where $a_c \neq 0$ holds. It can easily be seen that L_2 can only vanish if M_1, M_4, M_5 are collinear, which yields a contradiction. Therefore $L_1 = 0$ has to hold, which can only be the case for $A_4 = B_4 = 0$ ($\Rightarrow rk(\mathbf{J}) = 7$). This implies $B_r = 0$ and $B_c = \pm i$, which shows that \mathbf{s} is a straight cubic circle. \square

Theorem 10. *The conic \mathbf{q} of a pentapod of Type 2 with self-motions has to be a circle and the line \mathbf{g}_1 is orthogonal to its carrier plane (= degenerated case of a straight cubic circle).*

PROOF: The proof can exactly be done as for Theorem 9 under consideration of $a_4 = 0$, $C_5 = 0$ and that no two platform anchor points can coincide beside m_1 and m_4 .

It should only be noted that in this case L_2 vanishes for $M_1 = M_5$, which yields a contradiction to the properties of Type 2.⁸ Therefore we remain again with the solution $A_4 = B_4 = 0$, which implies $B_r = 0$ and $B_c = \pm i$. This proves the theorem. \square

Theorem 11. *The cubic s^* of a pentapod of Type 5 with self-motions has to be an irreducible straight cubic circle.*

PROOF: The proof of this theorem can be done in a similar fashion as those of Theorems 9 and 10. In order to streamline the presentation the proof of Theorem 11 is given in Appendix C. \square

Due to the Theorems 4,8,9,10,11 the condition of Duporcq [13], that the centers of the spheres have to be located on a straight cubic circle, is valid for non-planar pentapods of \mathcal{P} . Note that Duporcq also mentioned explicitly the special cases of Type 2 and Type 5 beside the general case of Type 1. Therefore it remains to show if this so-called Duporcq condition is already sufficient for the existence of a self-motion. This is done in the next section.

5.3. Sufficiency of the Duporcq condition

The sufficiency is proven separately for the Types 1,2,5. Moreover in each of the three proofs also the leg parameters for a self-motion are given.

Theorem 12. *A pentapod of Type 5 fulfilling the Duporcq condition has a 1-parametric set of self-motions (over \mathbb{C}). With respect to the coordinatization used in the proof of Theorem 11 (under consideration of $A_4 = B_4 = 0$ and $B_2 = i$) the leg parameters are given by:*

$$w = \frac{C_5}{a_5}, \quad p_2 = -\frac{(a_3 - a_5)(A_5 - iB_5)}{a_5}, \quad p_3 = -\frac{(a_2 - a_5)(A_5 + iB_5)}{a_5}, \quad (46)$$

and the following condition remains in R_1 and R_5 :

$$(a_5^2 + B_5^2 + C_5^2)(a_2 + a_3 - a_5) + (R_1^2 - R_5^2 - a_2a_5 - a_3a_5 + a_5^2)a_5 = 0. \quad (47)$$

PROOF: We use the same coordinatization as given in the first two paragraphs of the proof of Theorem 11 under consideration of $A_4 = B_4 = 0$ and $B_2 = i$. We distinguish two cases:

1. $C_5 - a_5w \neq 0$: Under this assumption we can solve $\Lambda_1, \Omega_2, \Omega_3, \Delta_4, \Lambda_5, \Phi_3$ for $x_3, y_0, y_1, y_2, y_3, n_0$. Plugging the obtained expressions into Φ_i yields the equations Φ_i^* in x_0, x_1, x_2 for $i = 1, 2$. Now we compute the resultant Ξ of the numerator of Φ_1^* (which is quadratic in x_0, x_1, x_2) and the numerator of Φ_2^* (which is quartic in x_0, x_1, x_2) with respect to x_2 . Ξ factors into $x_0^4 N[11464]$, where N is quartic in x_0, x_1 . As $x_0 \neq 0$ has to hold (cf. Section 4.1), N has to be fulfilled identically. Therefore we denote the coefficient of $x_0^i x_1^j$ of N by N_{ij} . Then N_{04} factors into $16E_1^2 E_2^2$ with

$$E_1 = (a_3 - a_5)(A_5 - iB_5) + a_5 p_2, \quad E_2 = (a_2 - a_5)(A_5 + iB_5) + a_5 p_3. \quad (48)$$

W.l.o.g. we can set E_1 equal to zero and solve it for p_2 . Then the numerator of N_{13} factors into $32a_3(a_3 - a_5)(A_5 - iB_5)(C_5 - a_5w)^2 E_2^2$. Therefore $E_2 = 0$ has to hold, which can be solved for p_3 w.l.o.g.. Now the numerator of N_{22} splits up into

$$64(a_5^2 + B_5^2)a_2a_3(a_2 - a_5)(a_3 - a_5)(C_5 - a_5w)^4, \quad (49)$$

which cannot vanish without contradiction.

⁸In this case the cubic of base points splits up into three lines; one real and two conjugate complex ones, which intersect each other in $M_1 = M_5$ (complex version of Type 4).

2. $w = C_5/a_5$: We can solve the equations $\Lambda_1, \Omega_2, \Omega_3, \Omega_4, \Lambda_5$ for x_3, y_1, y_2, y_3, n_0 and plug the obtained expressions into Φ_i , which yields quadratic equation Φ_i^* in x_0, x_1, x_2, y_0 for $i = 1, 2, 3$. Now then numerator of Φ_1^* and Φ_3^* do not depend on y_0 in contrast to the numerator of Φ_2^* (coefficient of y_0^2 equals $4a_5^2$). Therefore we compute the resultant Ξ of the numerators of Φ_1^* and the numerator of Φ_3^* with respect to x_2 . Ξ factors into $x_0^2 a_5^2 N$ [208], where N is quadratic in x_0, x_1 . We denote the coefficient of $x_0^i x_1^j$ of N again by N_{ij} . In the following we show that $E_1 = 0$ and $E_2 = 0$ has to hold:

N_{02} factors into $4a_5^2 E_1 E_2$, which can be solve w.l.o.g. for p_2 . Then N_{11} splits up into

$$-2a_5 E_2 \left[a_3(A_5^2 + B_5^2) + (a_2 + a_3 - a_5)C_5^2 + (1 - a_2 - a_3)a_5^2 + (R_1^2 - R_5^2)a_5 - (A_5 - iB_5)a_5 p_3 \right]. \quad (50)$$

Either $E_2 = 0$ holds and we are done or the last factor vanishes. In the latter case we can compute p_3 w.l.o.g.. Then N_{20} factors into $(C_5^2 - a_5^2)E_2^2$. For $C_5 = \pm a_5$ the condition Φ_1 equals $x_1^2 + x_2^2 = 0$, which cannot yield a real self-motion. Therefore $E_2 = 0$ has to hold.

Summed up we have proven that $E_1 = 0$ and $E_2 = 0$ have to hold. These equations can be solved for p_2, p_3 w.l.o.g.. Plugging the obtained expressions into N shows that only the condition given in Eq. (47) remains. This condition can always be solved for R_5^2 and the self-motion is again obtained by back-substitution, which finishes the proof of the sufficiency. \square

Theorem 13. *A pentapod of Type 1 fulfilling the Duporcq condition has a 1-parametric set of self-motions (over \mathbb{C}). With respect to the coordinatization used in the proof of Theorem 9 (under consideration of $A_4 = B_4 = 0$ and $B_2 = i$) the leg parameters are given by:*

$$p_2 = -\frac{A_5(a_2a_3 - a_4^2) - i(a_2a_3 - a_4^2)B_5}{(a_3 - a_4)^2}, \quad p_3 = -\frac{A_5(a_2a_3 - a_4^2) + i(a_2a_3 - a_4^2)B_5}{(a_2 - a_4)^2}, \quad p_4 = \frac{C_5(a_2a_3 - a_4^2)}{(a_2 - a_4)(a_3 - a_4)}, \quad (51)$$

and the following condition remains in R_1 and p_5 :

$$(a_2 - a_4)^2(a_3 - a_4)^2 \left[2(a_2a_3 - a_4^2)p_5 - (a_2 + a_3 - 2a_4)R_1^2 - (2a_2a_3 - a_2a_4 - a_3a_4)a_4 \right] + (a_2a_3 - a_4^2)^2(a_2 + a_3 - 2a_4)(A_5^2 + B_5^2 + C_5^2) = 0. \quad (52)$$

Theorem 14. *A pentapod of Type 2 fulfilling the Duporcq condition has a 1-parametric set of self-motions (over \mathbb{C}). With respect to the coordinatization used in the proof of Theorem 13 (under consideration of $a_4 = C_5 = 0$) the leg parameters are given by:*

$$p_2 = -\frac{a_2(A_5 - iB_5)}{a_3}, \quad p_3 = -\frac{a_3(A_5 + iB_5)}{a_2}, \quad p_4 = 0, \quad (53)$$

and the following condition remains in R_1 and p_5 :

$$(A_5^2 + B_5^2)(a_2 + a_3) + 2a_2a_3p_5 - R_1^2(a_2 + a_3) = 0. \quad (54)$$

PROOF OF THEOREM 13 AND THEOREM 14: The proofs of these theorems can be done in a fashion similar to the one of Theorem 12. For this reason and in order to streamline the presentation the corresponding proofs are given in Appendix D and Appendix E, respectively. \square

Note that the proof of the sufficiency in the Theorems 12, 13 and 14 was only done over \mathbb{C} ; i.e. the self-motion has not to be real.⁹ Arguments of reality were only used to exclude some special cases.

This can best be seen for the self-motions obtained for Type 5, as they belong to the class of Borel-Bricard motions. These are the only non-trivial motions where all points of the moving space have spherical trajectories (cf. [8, Chapter VI]; see also [29]). Note that this special case was also discussed in detail by Krames [10, Section 5]. In this case Φ_1^* equals $x_0^2(C_5^2 - a_5^2) + a_5^2(x_1^2 + x_2^2) = 0$. This already shows the following result:

⁹In contrast the conditions given in Theorem 1 and those for the cases (α, β, γ) of Section 1 are even sufficient for the existence of real self-motions.

Corollary 3. *The 1-parametric set of self-motions given in Theorem 12 is real if $|C_5| < |a_5|$ holds; complex otherwise.*

Until now we are not able to give a corresponding easy characterization for designs of Type 1 and 2 with real/complex self-motions. But the following two examples prove that real self-motions exist:

Example 1. *This example of a pentapod of Type 1 with a real self-motion is based on the formulas of $\Lambda_1, \Omega_2, \Omega_3, \Omega_4, \Pi_5$ given in the proof of Theorem 9. The geometry of the pentapod is determined by:*

$$a_2 = B_2 = i, \quad a_3 = B_3 = -i, \quad a_4 = 2, \quad A_4 = B_4 = 0, \quad A_5 = B_5 = C_5 = 1. \quad (55)$$

For the leg-parameters:

$$R_1 = \sqrt{3}, \quad p_2 = -\frac{3}{25} - \frac{21}{25}i, \quad p_3 = -\frac{3}{25} + \frac{21}{25}i, \quad p_4 = -\frac{3}{5}, \quad p_5 = \frac{46}{75}, \quad (56)$$

which are in accordance with Theorem 13, the pentapod has the following self-motion (under consideration of $x_0 = 1$; cf. Eq. (21)):

$$x_1 = \frac{7}{4}t^2 - \frac{7}{5}t - \frac{161}{300} \mp \frac{T}{300}, \quad x_2 = \frac{1}{4}t^2 - \frac{1}{5}t - \frac{23}{300} \pm \frac{7T}{300}, \quad x_3 = t, \quad (57)$$

$$y_1 = -\frac{1}{4}t^2 + \frac{1}{5}t + \frac{59}{300} \mp \frac{7T}{300}, \quad y_2 = \frac{7}{4}t^2 - \frac{7}{5}t - \frac{413}{300} \mp \frac{T}{300}, \quad y_3 = -2t + \frac{3}{5}, \quad (58)$$

with $T = \sqrt{-(75t^2 - 30t - 41)(75t^2 - 90t + 31)}$. Both branches (upper and lower one) are real for $t \in [t^-, t^+]$ with

$$t^- = \frac{1}{5} - \frac{2}{15}\sqrt{33}, \quad t^+ = \frac{1}{5} + \frac{2}{15}\sqrt{33}. \quad (59)$$

For this example we also show how to compute the finite base anchor point \mathbf{M} (with coordinates $(A, B, C)^T$), the finite platform anchor point \mathbf{m} (with coordinates $(a, 0, 0)^T$) and the leg length R of a further leg. Its corresponding sphere condition Λ has to be a linear combination of the given equations $\Lambda_1, \Omega_2, \Omega_3, \Omega_4, \Pi_5$; i.e.

$$\mu_1\Lambda_1 + \mu_2\Omega_2 + \mu_3\Omega_3 + \mu_4\Omega_4 + \mu_5\Pi_5 - \Lambda = 0 \quad (60)$$

for any choice of $n_0, x_0, \dots, x_3, y_0, \dots, y_3$. Therefore their nine coefficients imply nine equations in the ten unknowns $\mu_1, \dots, \mu_5, R, A, B, C, a$. This system has the following solution (in dependence of a):

$$\begin{aligned} \mu_1 &= 1, \quad \mu_2 = A + Bi, \quad \mu_3 = A - Bi, \quad \mu_4 = 2C, \quad \mu_5 = 2a, \\ R^2 &= \frac{(3a^2 - 8a + 9)(25a^4 - 64a^3 + 146a^2 - 136a + 100)}{75(a^2 + 1)(a - 2)^2}, \end{aligned} \quad (61)$$

with

$$A = \frac{a(a-1)}{a^2+1}, \quad B = \frac{a(a+1)}{a^2+1}, \quad C = \frac{a}{a-2}. \quad (62)$$

The last equation gives the bijection σ between points \mathbf{m} of \mathbf{p} and points \mathbf{M} of the irreducible straight cubic circle \mathbf{s} (cf. Eq. (16)).

In Fig. 5a the trajectories of the platform anchor points \mathbf{m}_1 ($a = 0$), \mathbf{m}_6 ($a = 1$), \mathbf{m}_7 ($a = 3$), \mathbf{m}_8 ($a = -1$) and \mathbf{m}_9 ($a = -2$) are displayed for the upper branch of the self-motion. \diamond

Example 2. *This example of a pentapod of Type 2 with a real self-motion is based on the formulas of $\Lambda_1, \Omega_2, \Omega_3, \Omega_4, \Pi_5$ given in the proof of Theorem 9 under consideration of $a_4 = 0$ and $C_5 = 0$. The geometry of the pentapod is determined by:*

$$B_2 = i, \quad B_3 = -i, \quad a_2 = 1 + i, \quad a_3 = 1 - i, \quad A_4 = B_4 = 0, \quad A_5 = B_5 = 1. \quad (63)$$

For the leg-parameters:

$$R_1 = 2, \quad p_2 = -1 - i, \quad p_3 = -1 + i, \quad p_4 = 0, \quad p_5 = 1, \quad (64)$$

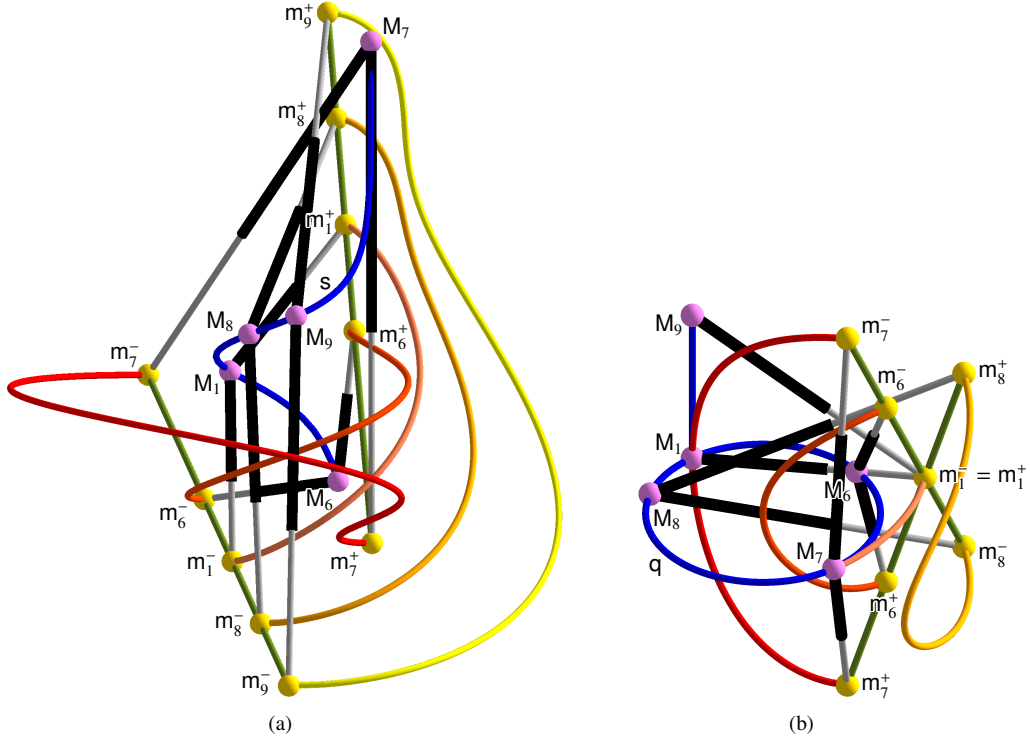


Figure 5: Trajectories of the upper branch of the self-motion, where the starting pose at time t^- and the end pose at time t^+ are illustrated: (a) Example 1 and (b) Example 2. Animations of these two self-motions can be downloaded from the homepage of the first author (<http://www.geometrie.tuwien.ac.at/nawratil>).

which are in accordance with Theorem 14, the pentapod has the following self-motion (under consideration of $x_0 = 1$; cf. Eq. (21)):

$$x_1 = -\frac{1}{2}t^2, \quad x_2 = \pm \frac{T}{2}, \quad x_3 = t, \quad y_1 = \frac{1}{2}t^2 + 1 \mp \frac{T}{2}, \quad y_2 = -\frac{1}{2}t^2 - 1 \mp \frac{T}{2}, \quad y_3 = 0, \quad (65)$$

with $T = \sqrt{-t^4 - 4t^2 + 4}$. Both branches (upper and lower one) are real for $t \in [t^-, t^+]$ with

$$t^- = -\sqrt{2\sqrt{2}-2}, \quad t^+ = \sqrt{2\sqrt{2}-2}. \quad (66)$$

Analogous considerations as in Example 1 show the following bijection σ between points m of $\mathfrak{p} \setminus \{m_1\}$ and points M of the circle $\mathfrak{q} \setminus \{M_1\}$:

$$A = \frac{a(a-2)}{a^2-2a+2}, \quad B = \frac{a^2}{a^2-2a+2}, \quad C = 0. \quad (67)$$

The point m_1 is mapped to the line $[M_1, M_9]$, which equals the z -axis of Σ_0 .

In Fig. 5b the trajectories of the platform anchor points $m_1 = m_9$ ($a = 0$), m_6 ($a = 1$), m_7 ($a = 2$) and m_8 ($a = -1$) are displayed for the upper branch of the self-motion. \diamond

Remark 6. Further examples of pentapods of Type 1 and Type 2 are implied by a remarkable motion, where all points of a hyperboloid, which carries two reguli of lines, have spherical trajectories. This well-studied motion is also known as BBM-II motion in the literature (e.g. [30]). It is known (cf. [30, page 24] and [31, page 188]) that the

corresponding sphere centers of lines, belonging to one regulus¹⁰, are located on irreducible straight cubic circles, which imply examples of self-motions of Type 1. Note that there also exist degenerated cases where the hyperboloid splits up into two orthogonal planes, which imply examples of self-motions of Type 2. \diamond

6. Conclusions for practical applications

We introduced a novel kinematic mapping for pentapods with linear platform (cf. Section 4.1), which can be used for an efficient solution of the direct kinematics problem. Beside this achievement we listed all pentapods with linear platform which

- ★ are architecturally singular (cf. Corollary 1),
- ★ possess self-motion (over \mathbb{C}) without having an architecture singularity.

The latter are either the designs (α, β, γ) given in Section 1, pentapods of Types 1,2,5 fulfilling the Duporcq condition (cf. Section 5) or the manipulator given in Theorem 1.

Clearly, architecturally singular pentapods are not suited for practical application and therefore engineers should be aware of these designs. The usage of pentapods with self-motions within the design process is a double-edged sword; on the one side they should be avoided for reasons of safety¹¹ and on the other side they have a simplified direct kinematics for the following reason:

As the bonds are independent of the set of leg lengths (cf. Section 4.2), they always appear as solution of the direct kinematics problem even though the given set R_1, \dots, R_5 does not cause a self-motion of the manipulator. Recall that a bond of a self-motion corresponds to a complex configuration on the boundary (cf. Remark 5). Therefore four solutions (= two conjugate complex bonds of multiplicity 2; cf. Remark 5) of the direct kinematics of a pentapod with linear platform possessing a self-motion are always located on the boundary of \mathcal{I} . This yields the following corollary:

Corollary 4. *A pentapod with linear platform possessing a self-motion (over \mathbb{C}) can have a maximum of four real configurations (instead of generically 8, cf. Theorem 7), if the given set of leg lengths does not imply a self-motion. The direct kinematic problem of these pentapods reduces to the solution of a polynomial of degree 4.*

As this quartic equation can be solved explicitly, these pentapods seem to be of special interest for practical application. We demonstrate this result on the basis of the following example:

Example 3. *Continuation of Example 1: We consider the pentapod with platform anchor points m_1, m_6, \dots, m_9 and base anchor points M_1, M_6, \dots, M_9 of Example 1 (see Fig. 5a). But now we want to solve the direct kinematics problem of this pentapod of Type 1 for the following given set of leg lengths:*

$$R_1 = 2, \quad R_6 = 1, \quad R_7 = 5, \quad R_8 = 3, \quad R_9 = 4, \quad (68)$$

which does not cause a self-motion. We can solve the corresponding system of equations $\Lambda_1, \Lambda_6, \dots, \Lambda_9$ for n_0, y_0, y_1, y_2 and y_3 . Moreover we can set $x_0 = 1$. Now Φ_3 of Eq. (22) is only linear in x_1 and x_2 and we can solve it for x_1 . Then Φ_1 and Φ_2 are only quadratic in x_2 , and therefore the resultant of these two expressions with respect to x_2 yields:

$$4316636297 + 69486876480x_3 + 241133479200x_3^2 - 291209472000x_3^3 + 76425120000x_3^4 = 0, \quad (69)$$

which is only of degree 4 in x_3 . \diamond

But it is even possible to use this advantage of self-motions without any risk (cf. footnote 9), by designing pentapods with linear platform, which only have complex self-motions. In the following we list two sets of such designs:

- (I) Pentapods of Type 5 with $|C_5| \geq |a_5|$ (cf. Corollary 3).

¹⁰The corresponding sphere centers of lines belonging to the other regulus are again located on a line (cf. [30, page 24]), which imply architecturally singular pentapod designs.

¹¹A self-motion is dangerous because it is uncontrollable and thus a hazard to man and machine.

- (II) We can also solve the planar case (cf. Section 2) with the bond based approach used for the non-planar one. This study shows, that a planar pentapod of \mathcal{P} (cf. Definition 1) has a bond if and only if the vertex V (cf. proof of Theorem 1) is an ideal point. Moreover the second necessary condition implied by the theory of bonds is only fulfilled if the affine relation (AR) holds¹².

Remark 7. Note that the condition (AR) equals the linear constraint given in [1], where the observation was reported that these planar pentapods with linear platform only possess a maximum of four real solutions of the direct kinematics problem (without giving an explanation for this behavior). This problem can even be solved quadratically as the solutions are symmetric with respect to the base plane (cf. [1]). Therefore this also holds for the design (γ) and for the designs (α, β) under the extra condition of a planar base. \diamond

The constraint (AR) is even sufficient for the existence of a self-motion (over \mathbb{C}). Now we design the pentapods in a way that the distance between the parallel lines $[M_i, V]$ and $[M_j, V]$, which are fibers of the affinity κ , is

- equal or less than the distance $\text{dist}(m_i, m_j)$ between their images: This yields the pentapods characterized in Theorem 1, which all have real self-motions (cf. footnote 8).
- greater than the distance $\text{dist}(m_i, m_j)$ between their images: Then the line p cannot be oriented that $\text{dist}(M'_i, M'_j) = \text{dist}(m'_i, m'_j)$ holds, which already shows that this design-set (II) is free of real self-motion.

Therefore the authors recommend engineers to design pentapods with linear platform within the set (I) in the non-planar case and within the set (II) in the planar one, respectively. For reasons of completeness we also give the following corollary:

Corollary 5. *The following pentapods with linear platform, which do not possess a self-motion, have a maximum of six real configurations (instead of generically 8, cf. Theorem 7):*

1. *The irreducible cubic s of a pentapod of Type 1 is a cubic ellipse located on a cylinder of revolution.*
2. *The conic q of a pentapod of Type 2 belongs to a cylinder of revolution, where one generator is the line g_1 .*
3. *A pentapod of Type 5 with one of the following two properties:*
 - (a) *Either the base anchor points of possible leg-replacements are located on an irreducible cubic ellipse s^* on a cylinder of revolution.*
 - (b) *or it has the following design (under consideration of footnote 1): $m_2 = m_3$, $m_4 = m_5$ and $[M_2, M_3]$ is parallel to $[M_4, M_5]$.*
4. *Planar pentapod of \mathcal{P} , where the associated point V is an ideal point (cf. Fig. 3c).*

The direct kinematic problem of the listed pentapods reduces to the solution of a polynomial of degree 6.

PROOF: Items 1, 2 and 3(a) are a direct consequence of the Theorems 6, 7 and 5, respectively. Item 4 follows from the above given discussion of design-set (II) and item 3(b) can be seen as its corresponding non-planar case (cf. Fig. 3c). In the latter case the cubic splits up into 3 parallel (but non-planar) lines. This case is hidden in item 2(a) of the proof of Theorem 5.

Finally, it should be noted that the direct kinematics problem is only cubic for item 4 as the pentapod is planar (cf. Remark 7). \square

We want to close the paper by referring to [32] where the remaining problem of characterizing designs of Type 1 and Type 2 without real self-motions is discussed, as they imply further "save" designs with a closed form solution.

¹²In this case the bond (and its conjugate) are singular points of the 5-fold \mathcal{I} (cf. Remark 5).

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Appendix A: Proof of Theorem 7

We can start with the same set of equations as in the proof of Theorem 6, but we have the extra conditions $m_1 = m_4$ and that M_1, M_2, M_3, M_5 are located in a plane, which does not contain M_4 . Due to the properties of Type 2 not all base points are coplanar ($\Rightarrow B_2C_3C_5 \neq 0$) and no two platform anchor points can coincide beside m_1 and m_4 ($\Rightarrow a_1 = a_4 = 0$). Therefore we can express the coplanarity of M_1, M_2, M_3, M_5 as

$$A_3 = \frac{A_2B_3C_5 - A_2B_5C_3 + B_2C_3A_5}{B_2C_5}. \quad (70)$$

We set $x_0 = 0$ and start the computation of the bonds: We can solve the equations $\Lambda_1, \Lambda_2, \Lambda_3, \Omega_4, \Pi_5$ for y_0, y_1, y_2, y_3, n_0 w.l.o.g.. Now the numerator of Γ_4 factors into x_1F with F of Eq. (36). Therefore we have to distinguish two cases:

1. $x_1 \neq 0$: In this case $F = 0$ has to hold, which can be solve w.l.o.g. for x_3 . Then the numerator of Γ_6 factors into:

$$x_2a_3(x_2B_2 + x_1A_2)(B_3C_5 - B_5C_3 + B_2C_3 - B_2C_5). \quad (71)$$

The last factor cannot vanish, as otherwise the base anchor points M_2, M_3, M_5 are collinear, a contradiction. Therefore we remain with two cases:

- a. $x_2 = -x_1A_2/B_2$. Now the numerator of Γ_1 factors into:

$$x_1^2 \left[A_2^2(B_5^2 + C_5^2) - 2A_2A_5B_2B_5 + B_2^2(A_5^2 + C_5^2) \right]. \quad (72)$$

The discriminant with respect to A_2 equals $-C_5^2(A_5^2 + B_5^2 + C_5^2)$ and therefore we get a contradiction.

- b. $x_2 = 0$: In this case the numerator of Γ_1 factors into $x_1^2[C_5^2 + (A_2 - A_5)^2]$ which can also not vanish without contradiction.
2. $x_1 = 0$: This case is exactly the same as the one discussed in item 2 of the proof of Theorem 6, which already yields the result. \square

Appendix B: Proof of Theorem 8

We can make leg-replacements such that $m_1 = m_2 = P_1$, $m_3 = m_4 = P_2$, $M_1, M_2 \in L_1$ and $M_3, M_4 \in L_2$. Moreover we can choose M_2 and M_3 as ideal points of L_1 and L_2 , respectively. Therefore the point pairs (M_i, m_i) determine sphere conditions Λ_i for $i = 1, 4$ and Darboux condition Ω_i for $i = 2, 3$. Moreover we can assume that M_1 is located in the Darboux plane of (M_2, m_2) and that M_4 is located in the Darboux plane of (M_3, m_3) . Finally we can assume that m_5 is the ideal point of the line p . Therefore the point pairs (M_5, m_5) determines a Mannheim condition Π_5 .

W.l.o.g. we choose the fixed frame Σ_0 that M_1 equals its origin and that M_2 and M_3 are located in the xy -plane symmetric with respect to the x -axis. Therefore the directions of M_2 and M_3 are given by $(1, B_2, 0)$ and $(1, -B_2, 0)$, respectively. Moreover we can define the moving frame Σ in a way that m_1 is its origin. With respect to these coordinate systems our conditions can be written as:

$$\begin{aligned} \Omega_2 : & \quad y_1 + B_2y_2 = 0, \\ \Omega_3 : & \quad (A_4 - B_2B_4)x_0 + a_3x_1 - a_3B_2x_2 + y_1 - B_2y_2 = 0, \\ \Pi_5 : & \quad p_5x_0 + A_5x_1 + B_5x_2 + C_5x_3 + y_0 = 0, \end{aligned} \quad (73)$$

where $(p_5, 0, 0)^T$ are the coordinates of the intersection point of the Mannheim plane and the x -axis of Σ . The equations of Λ_i for $i = 1, 4$ are given in Eq. (20) under consideration of $A_1 = B_1 = C_1 = a_1 = 0$ and $a_4 = a_3$.

Now we set $x_0 = 0$ and prove that no bonds can exist. W.l.o.g. we can solve $\Lambda_1, \Omega_2, \Omega_3, \Pi_5$ for n_0, y_0, y_1, y_2 . Then the numerator of Γ_4 can only vanish in the following two cases:

1. $x_1 = B_2x_2$: The numerator of Γ_2 implies $y_3 = 0$. Now the numerator of Λ_4 equals:

$$a_4 [x_2(B_2A_4 - B_2A_5 + B_4 - B_5) + (C_4 - C_5)x_3]. \quad (74)$$

We distinguish two cases:

- (a) $C_4 \neq C_5$: Under this assumption we can solve the last factor for x_3 . Then x_2^2 factors out from the numerator of Γ_1 and we remain with only one condition, which is quadratic with respect to A_5 . The corresponding discriminant equals $-(C_4 - C_5)^2(B_2^2 + 1)$ and therefore no solution exists.
- (b) $C_4 = C_5$: In this case Λ_4 can only vanish for $x_2 = 0$ as $B_2A_4 - B_2A_5 + B_4 - B_5 = 0$ implies the collinearity of M_3, M_4, M_5 , a contradiction. Then Γ_1 cannot vanish without contradiction.

2. $x_1 = -B_2x_2$: Now the numerator of Γ_6 equals $-x_2(x_3a_4 + y_3)$. Therefore we have to distinguish two cases:

- (a) $x_2 = 0$: Now Γ_1 and Γ_2 imply $x_3 = y_3 = 0$, a contradiction.
- (b) $y_3 = -x_3a_4$: Then Λ_4 factors into

$$a_4 [x_2(B_2A_5 - B_5) - C_5x_3]. \quad (75)$$

We distinguish two cases:

- i. $C_5 \neq 0$: Under this assumption we can solve the last factor for x_3 . Then x_2^2 factors out from the numerator of Γ_1 and we remain with only one condition, which is quadratic with respect to A_5 . The corresponding discriminant equals $-C_5^2(B_2^2 + 1)$ and therefore no solution exists.
- ii. $C_5 = 0$: In this case Λ_4 can only vanish for $x_2 = 0$ as $B_2A_5 - B_5 = 0$ implies the collinearity of M_1, M_2, M_5 , a contradiction. Then Γ_1 cannot vanish without contradiction. \square

Appendix C: Proof of Theorem 11

Due to Theorem 5 the irreducible cubic \mathbf{s}^* has three pairwise distinct points at infinity, which are denoted by M_2, M_3, M_4 . Note that M_4 is real and that M_2, M_3 are conjugate complex; i.e. $\overline{M_2} = M_3$. The corresponding platform anchor points are denoted by m_2, m_3, m_4 where $\overline{m_2} = m_3$ holds. Therefore we get two Darboux conditions Ω_i implied by the point pairs (M_i, m_i) for $i = 2, 3$. Moreover the point pair (M_4, m_4) implies the angle condition \angle_4 . The pentapod is completed by two sphere conditions Λ_j , which are determined by the two finite points M_j and m_j for $j = 1, 5$.

We choose the fixed frame Σ_0 that M_1 is the origin and that M_2 and M_3 are located in the xy -plane in direction $(1, B_2, 0)$ and $(1, \overline{B_2}, 0)$, respectively. As M_2, M_3, M_4 cannot be collinear, M_4 is the ideal point in direction of $(A_4, B_4, 1)$. Moreover we can define the moving frame Σ in a way that m_1 is its origin. With respect to these coordinate systems our conditions can be written as:

$$\Omega_i : p_i x_0 + a_i x_1 + a_i \overline{B_i} x_2 + y_1 + \overline{B_i} y_2 = 0, \quad \angle_4 : w x_0 + A_4 x_1 + B_4 x_2 + x_3 = 0 \quad (76)$$

where $(p_i, 0, 0)^T$ for $i = 2, 3$ are the coordinates of the intersection point of the Darboux plane and the x -axis of Σ_0 and w denotes $\arccos(\varphi)$. The equations of Λ_j is given in Eq. (20) under consideration of $A_1 = B_1 = C_1 = a_1 = 0$ for $j = 1, 5$.

In the following we substitute $B_2 = B_r + iB_c$ and $a_2 = a_r + ia_c$ with $B_r, B_c, a_r, a_c \in \mathbb{R}$ and $B_c a_c \neq 0$. Then we set $x_0 = 0$ and start the computation of bonds: We can solve the equations $\Lambda_1, \Omega_2, \Omega_3, \angle_4, \Lambda_5$ for x_3, y_1, y_2, y_3, n_0 w.l.o.g.. Now the numerator of Γ_4 can only vanish without contradiction for $x_1 = (-B_r \pm iB_c)x_2$. We only discuss the upper sign as the conjugate solution can be done analogously. Then the numerator of Γ_6 can be solved for y_0 w.l.o.g.. From the numerator of Γ_1 we can factor out x_2^2 and we remain with only one condition. From its imaginary part and real part we can compute B_r and B_c given in Eq. (42).

Then we compute again the 8×9 matrix \mathbf{J} with respect to the obtained bond. For the necessary condition $rk(\mathbf{J}) < 8$ the determinants of all 8×8 submatrices of \mathbf{J} have to vanish. The numerator of the determinant of the 8×8 submatrix of \mathbf{J} , obtained by removing the column stemming from the partial derivative with respect to x_2 , factors into $(ia_r + a_c - ia_5)(ia_r + a_c)(A_4^2 + B_4^2 + 1)L_1 L_2$ with L_1 and L_2 of Eq. (45). Analogous arguments as in the proof of Theorem 9 shows that \mathbf{s}^* has to be a straight cubic circle. \square

Appendix D: Proof of Theorem 13

We use the same coordinatization as given in the first three paragraphs of the proof of Theorem 9 under consideration of $A_4 = B_4 = 0$ and $B_2 = i$.

W.l.o.g. we can solve the equations $\Lambda_1, \Omega_2, \Omega_3, \Omega_4, \Pi_5$ for y_0, y_1, y_2, y_3, n_0 . Plugging the obtained expressions into Φ_i yields quadratic equation Φ_i^* in x_0, \dots, x_3 for $i = 1, 2, 3$. Therefore Φ_i^* corresponds with a quadric in the homogeneous 3-space (spanned by x_0, \dots, x_3). In the following we want to determine the parameters R_1, p_2, p_3, p_4, p_5 in a way that these three quadrics have a curve in common. This can be done as follows:

We compute the resultant Ξ_k of Φ_i^* and Φ_j^* with respect to x_1 for pairwise distinct $i, j, k \in \{1, 2, 3\}$. Ξ_1, Ξ_2 and Ξ_3 are homogeneous quartic expressions in x_0, x_2, x_3 , but they are only quadratic with respect to x_2 . Therefore we eliminate this unknown by computing the resultant Υ_k of Ξ_i and Ξ_j for pairwise distinct $i, j, k \in \{1, 2, 3\}$. Then the greatest common divisor N of Υ_1, Υ_2 and Υ_3 has to vanish. It turns out that N has 411 terms and that it is homogeneous of degree 4 in x_0, x_3 . For a self-motion of the line \mathbf{p} the expression N has to be fulfilled independently of x_0, x_3 . Therefore we denote the coefficient of $x_0^i x_3^j$ of N by N_{ij} .

In the following we show that the three conditions $E_3 = 0, E_4 = 0, E_5 = 0$ have to be fulfilled with:

$$\begin{aligned} E_3 &= A_5(a_2a_3 - a_4^2) + p_2(a_3 - a_4)^2 - i(a_2a_3 - a_4^2)B_5, \\ E_4 &= A_5(a_2a_3 - a_4^2) + p_3(a_2 - a_4)^2 + i(a_2a_3 - a_4^2)B_5, \\ E_5 &= C_5(a_2a_3 - a_4^2) - p_4(a_2 - a_4)(a_3 - a_4). \end{aligned} \quad (77)$$

This can be seen as follows: $N_{04} = 0$ splits up into E_3E_4 . W.l.o.g. we can set E_3 equal to zero and solve it for p_2 . Then the numerator of N_{13} factors into $-2a_3(A_5 - iB_5)E_4E_5$. Therefore we have to distinguish two cases:

1. $E_4 = 0$: W.l.o.g. we can solve this equation for p_3 . Then the numerator of N_{22} splits up into $4a_2a_3(A_5^2 + B_5^2)E_5^2$. Therefore $E_5 = 0$ has to hold and we are done.
2. $E_5 = 0$: W.l.o.g. we can solve this equation for p_4 . Then the numerator of N_{22} and N_{31} splits up into:

$$a_3(A_5 - iB_5)E_4F_{22}[83], \quad a_3a_4C_5(A_5 - iB_5)E_4F_{31}[73]. \quad (78)$$

Either $E_4 = 0$ holds and we are done or $F_{22} = 0$ and $F_{31} = 0$ have to hold. In the latter case we compute $F_{22} + F_{31}$, which factors into $a_3(a_2 - a_4)^2(A_5 - iB_5)E_4$. Therefore again $E_4 = 0$ has to be fulfilled.

Summed up we have proven that $E_3 = 0, E_4 = 0, E_5 = 0$ have to hold. These equations can be solved for p_2, p_3, p_4 w.l.o.g.. Plugging the obtained expressions into N shows that only the condition given in Eq. (52) remains. We distinguish two cases:

- $a_2a_3 - a_4^2 \neq 0$: In this case Eq. (52) can always be solved for p_5 . Then the self-motion can be computed by back-substitution; i.e. we compute the common factor of Ξ_1, Ξ_2, Ξ_3 and solve it for x_2 (which only appears quadratic) and finally the common factor of Φ_1^*, Φ_2^* and Φ_3^* (which is linear in x_1) gives the self-motion.
- For the special case $a_2a_3 - a_4^2 = 0$ we get $R_1^2 = a_4^2$ from Eq. (52). Moreover if we set $a_2 = a_r + ia_c$ with $a_r, a_c \in \mathbb{R}$ and $a_c \neq 0$ the condition $a_2a_3 - a_4^2 = 0$ is equivalent with $a_r^2 + a_c^2 - a_4^2 = 0$, which can be solved for a_4 w.l.o.g.. Now it can easily be seen that $\Phi_1^*(a_r^2 + a_c^2) = \Phi_2^*$ holds. Therefore we only remain with the following condition beside Φ_1^* (which equals Φ_1 given in Eq. (22)):

$$\Phi_3^* : x_0^2 p_5 - (x_1^2 + x_2^2) a_r - x_3^2 \sqrt{a_r^2 + a_c^2} + x_0(x_1 A_5 + x_2 B_5 + x_3 C_5) = 0. \quad (79)$$

This finishes the proof of the sufficiency. □

Appendix E: Proof of Theorem 14

We start with the same set of equations as in the proof of Theorem 13 under consideration of $a_4 = 0$, $C_5 = 0$ and that no two platform anchor points can coincide beside m_1 and m_4 .

Moreover we can compute analogously the corresponding expression N , which has in this case only 204 terms. We denote the coefficient of $x_0^i x_3^j$ of N again by N_{ij} . In the following we show that the three conditions $E_6 = 0$, $E_7 = 0$ and $p_4 = 0$ have to be fulfilled with:

$$E_6 = a_2(A_5 - iB_5) + a_3p_2, \quad E_7 = a_3(A_5 + iB_5) + a_2p_3. \quad (80)$$

This can be seen as follows: $N_{04} = 0$ splits up into $a_2a_3E_6E_7$. W.l.o.g. we can set E_6 equal to zero and solve it for p_2 . Then the numerator of N_{13} factors into $2a_2^2(A_5 - iB_5)p_4E_7$. Therefore we have to distinguish two cases:

1. $E_7 = 0$: W.l.o.g. we can solve this equation for p_3 . Then the numerator of N_{22} splits up into $4a_2a_3(A_5^2 + B_5^2)p_4^2$. Therefore $p_4 = 0$ has to hold and we are done.
2. $p_4 = 0$: Now the numerator of N_{22} splits up into:

$$a_2(A_5 - iB_5)E_7 \left[a_2p_3(A_5 - iB_5) + R_1^2(a_2 + a_3) - a_2(A_5^2 + B_5^2 + 2a_3p_5) \right]. \quad (81)$$

If $E_7 = 0$ holds we are done. Therefore we express p_3 from the last factor, which can be done w.l.o.g.. Then N_{31} is fulfilled identically and we only remain with N_{40} . Its numerator factors into:

$$a_2R_1^2 \left[2a_2a_3p_5 + (a_2 + a_3)(A_5^2 + B_5^2 - R_1^2) \right]. \quad (82)$$

Now we have to distinguish two cases:

- (a) We can solve the last factor for p_5 w.l.o.g.. Then it can easily be checked that E_7 is fulfilled and we are done.
- (b) $R_1 = 0$: This condition implies a spherical self-motion of the line \mathbf{p} with center $M_1 = m_1$. Now a one-parametric set of legs with base anchor points on the circle \mathbf{q} (the corresponding platform anchor points on \mathbf{p} are given by σ^{-1}) can be attached without restricting the spherical self-motion. It can easily be seen that any two legs of this set already fix the pose of \mathbf{p} ; hence no real self-motion of \mathbf{p} exists.

Summed up we have proven that $E_6 = 0$, $E_7 = 0$ and $p_4 = 0$ have to hold. $E_6 = 0$, $E_7 = 0$ can be solved for p_2, p_3 w.l.o.g.. Plugging the obtained expressions into N shows that only the condition given in Eq. (54) remains. This condition can always be solved for p_5 and the self-motion is again obtained by back-substitution, which finishes the proof of the sufficiency. \square